1 Abouzaid's Generation Criterion, Jeff Hicks

1.1 Recall Generation Criterion

One of the problems in understanding the Fukaya category is to get an understanding of the objects in the category (as we do not have a very good understanding of what Lagrangians exist in a symplectic manifold.)
Our approach so far has been to pick out a certain class of Lagrangians, to show that these "split generate" the category. Abouzaid's generation criterion provides a way to do this.

Notation 1. *In this section:*

- *(M, ω) is a Liouville manifold*
- *$\mathcal{W}$ is the Wrapped Fukaya category of M.*
- *$\mathcal{B}$ is a full subcategory of M*

Theorem 1. *There is a map*

$$HH_*(\mathcal{B}) \to HH_*(\mathcal{W}) \to SH(M)$$

such that whenever the identity lies in the image, $\mathcal{B}$ split-generates $\mathcal{W}$.

Outline of Proof. There is an algebraic component and geometric component to this proof.
Algebraic Input: What we need to show is that a particular lagrangian $K \in Ob(\mathcal{W})$ is split generated by objects in $\mathcal{B}$. For any pair $L, L' \in \mathcal{B}$, there is a diagonal map

$$\Delta : HW^*(L, L') \to HW * (K, L') \otimes HW^*(L, K)$$

which is like a "dual" to the multiplication map. When we assembly these together to be a map of A_∞ modules, this is the map between

$$\Delta : \mathcal{B} \to \mathcal{Y}_K^l \otimes \mathcal{Y}_k^r$$

where these are the left and right $\mathcal{W}$-modules given by the Yoneda functor.
Whenever we have a morphism between bimodules, we get a morphism between Hochschild homology with coefficients in those bimodules. This gives us a map

$$HH_*(\Delta) : HH_*(\mathcal{B}, \mathcal{B}) \to HH_*(\mathcal{B}, \mathcal{Y}_K^l \otimes \mathcal{Y}_K^r)$$

There is an interpretation of Hochschild homology whenever the coefficients are taken in a product of a left and right module.

$$HH_*(\mathcal{B}, \mathcal{Y}_K^l \otimes \mathcal{Y}_k^r) = H^*(\mathcal{Y}_r^l \otimes_{\mathcal{B}} \mathcal{Y}_K^r)$$

This object comes with the multiplication map back down to the wrapped Floer cohomology of K, giving us the composition

$$HH_*(\mathcal{B}, \mathcal{B}) \xrightarrow{\Delta} HH^*(\mathcal{B}, \mathcal{Y}_K^l \otimes \mathcal{Y}_K^r) \simeq H^*(\mathcal{Y}_r^l \otimes_{\mathcal{B}} \mathcal{Y}_K^r) \xrightarrow{\mu} HW^*(K, K).$$

Lemma 1. *Whenever the identity of $HW^*(K, K)$ is in the image of this map, $\mathcal{B}$ split-generates K.*

This gives us an algebraic criterion for generation.
Geometric Input: There is a geometric interpretation of this map, which is the composition of the open-closed and closed-open maps. One interpretation of the Hochschild homology of the Fukaya category is given by the Symplectic cohomology, which is a count of pseudoholomorphic disks which have boundaries approaching Reeb orbits in the boundary.

- The Open-Closed map from the Hochschild homology to Symplectic Cohomology counts the number of punctured disks with internal puncture converging to a Reeb orbit, and boundary on the Hochschild chain.
- The Closed-Open map from the Symplectic Cohomology to the Wrapped Fukaya category counts punctured disks with one boundary point mapping to the Reeb orbit, and the other boundary point mapping to a chord.

$$HH_*(\mathcal{B},\mathcal{B}) \xrightarrow{H^*(\mathcal{OC})} SH^*(M) \xrightarrow{H^*(\mathcal{CO})} HW^*(K,K)$$

This again gives us a map from Hochschild homology of B to symplectic cohomology.

Lemma 2. *This is the same as the composition defined above.*

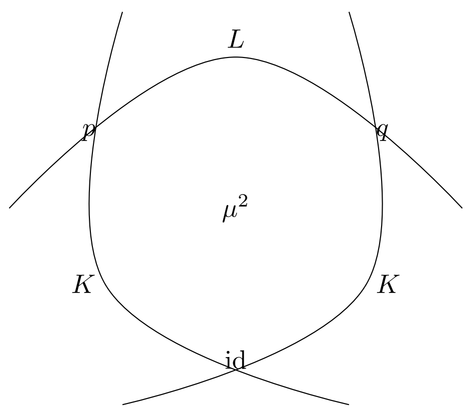
□

1.2 Algebraic Preliminaries

First, we are going to ask what algebraically we need to have a twisted complex split generate a particular Lagrangian K. We would need to show that K is a subobject of some twisted complex L. To exhibit K as a subobject of L, we need the following commutative diagram:

$$\begin{tikzcd} K \arrow[rr, "\mathrm{id}"] \arrow[dr] & & K \\ & L \arrow[ur] & \end{tikzcd}$$

If we were working with a single L, this would be the exhibition of a triangle that looks like this:

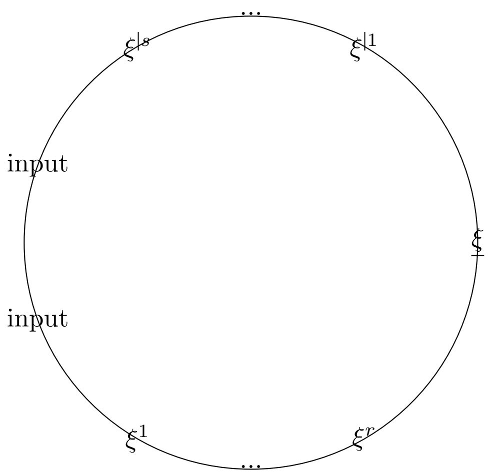


Because we are working with a twisted chain complex, the top edge can have some additional components. The existence of such morphisms is (roughly) the same as saying that $\mathrm{Hom}(K,\mathcal{B})\otimes\mathrm{Hom}(\mathcal{B},K) \to \mathrm{Hom}(K,K)$ hits the identity map. To make this precise you need the language of modules over A_∞ module, or Hochschild chains. To take into account that we are allowing the middle section to be a twisted complex, we are checking that the identity is in the image of:

$$HH^*(\mathcal{B}, \mathcal{Y}^l_K \otimes Y^r_K)) \xrightarrow{\mu} HW^*(K,K).$$

Here, the Hochschild chain component gives us the twisted complex.
However, we might think that this is hard to analyze without considering K, as both the domain and codomain depend on K. There is a way to remove this dependence in at least the domain by precomposing with a "comultiplication" map from $\mathcal{B} \xrightarrow{\Delta} \mathcal{Y}^l_K \otimes Y^r_K$. Recall that a map of A_∞ bimodules also preserves some

kind of commutivity of the structure maps, so we have to give a lot of morphism. This is given by taking a count of curves with $r+s+3$ punctures, arranged as :



Where the top punctures go to chords in $\{\mathcal{X}(L_{k-1},k)\}_{k=1}^r$, the bottom punctures go to chords $\{\mathcal{X}(L_{k-1},k)\}_{k=1}^s$, and the three remaining punctures go to

- A chord between $L_{|0}$ and L_0,
- The "output" chords in $CW^*(K,L_r)$ and $L_{|s},K$.

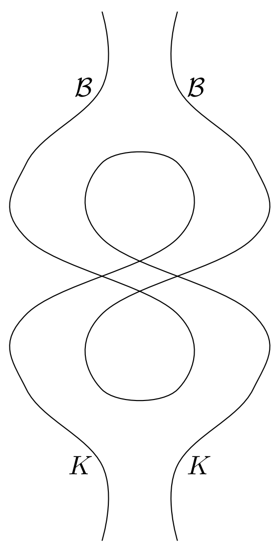
This produces a map A_∞ bi-module map

$$\Delta^{r|1|s}:\left(\bigotimes_{i=r}^{1} CW^*(L_{r-1},L_r)\right)\otimes CW^*(L_{|0},L_0)\otimes\left(\bigotimes_{k=1}^{s} CW^*(L_{|k},L_{|k-1})\right)\to CW^*(L_{|s},K)\otimes CW^*(K,L_r).$$

To actually define this map takes a substantial amount of work, to check that the moduli spaces that are in this count are well defined.
Putting this together gives us a composition

$$HH_*(\mathcal{B},\mathcal{B})\xrightarrow{\Delta} HH^*(\mathcal{B},\mathcal{Y}_K^l\otimes\mathcal{Y}_K^r)\simeq H^*(\mathcal{Y}_K^l\otimes_{\mathcal{B}}\mathcal{Y}_K^r)\xrightarrow{\mu} HW^*(K,K).$$

This composition amounts to taking a count of holomorphic disks that look like this:



We will want to express these disks geometrically.

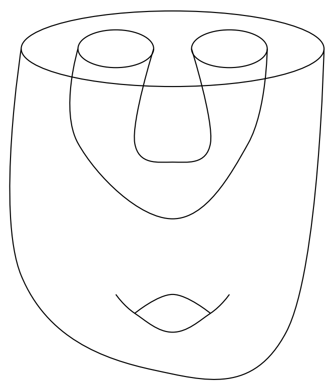
1.3 Symplectic cohomology

This can be expressed with symplectic cohomology. We give a quick overview here of the definition:
A *Reeb vector field* R_α on a contact manifold (Y, λ) such that

$$\iota_{R_\alpha} d\alpha = 0$$
$$\alpha(R_\alpha) = 1$$

- The generators are Reeb orbits in the contact boundary
- The differential is given by taking a count of holomorphic disks going between Reeb orbits

The picture you should have in mind is something like this:



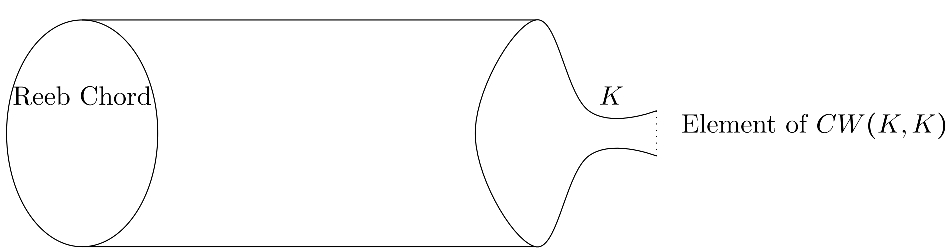
The differential counts the cylinders between these orbits. The moduli space cylinders satisfies a compactification by broken curves causing the differential to square to zero.

1.3.1 Identity

The identity component in symplectic homology is given by the Reeb orbits which bound holomorphic disks.

1.3.2 Open Closed and Closed Open maps

There is a map from symplectic cohomology to the Lagrangian cohomology which is given by counting disks with a single interior puncture and a single marked point on the boundary. The map is given by counting rigid holomorphic disks with the strip-like end going to the self Reeb chord of a Lagrangian K.

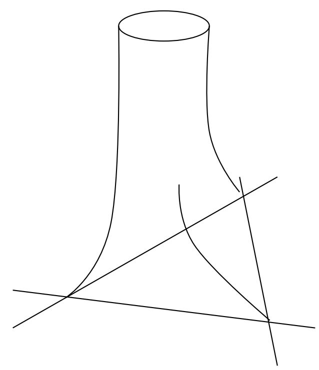


We call this the "closed-open map" $CO : SH(X) \to HW(K, K)$.

Claim 1. *The identity of $HW(K, K)$ is in the image of the open-closed map.*

Proof. The image of Reeb orbits which bound holomorphic disks is the identity in the Wrapped Fukaya category. These are cylinders attached to disks by a gluing argument. □

There is also a map from the Hochschild Homology of the Wrapped Fukaya category to the Symplectic Cohomology, given by counting disks of the following configuration:



This map gives us a "geometric deformation" of the Fukaya category by considering the disks which are removed by going through this divisor.
The space of open closed and closed open cylinders can be glued together into the space of all cylinders from cycles of Lagrangians in $\mathcal{B}$ to