An Exposition on Family Floer Theory

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Abstract

These are some notes for Berkeley's Mirror Symmetry Seminar 2017. In this survey, we outline the relation between Homological Mirror Symmetry to SYZ mirror Symmetry via the Family Floer theory. We also provide background sketches information on HMS, SYZ, and rigid analytic geometry. Primary sources for these notes include Abouzaid's papers on SYZ and Family Floer Theory [Abo14a] and [Abo14b], Hitchin's notes on the geometry of SYZ fibrations [Hit01], and [Con08]

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1 SYZ and HMS

The Strominger-Yau-Zaslow conjecture ([SYZ96]) gives us a geometric intuition for mirror symmetry. It conjectures that mirror symmetry is a relation of two different kinds of calibrated geometry, which can be interchanged by implementing a Fourier-transform on a family of special Lagrangian tori [LYZ00]. The SYZ formulation of mirror symmetry ties together geometry and topology in a way that explains predictions such as the interchange of Hodge numbers [Gro01] and forms the basis of the Gross-Seibert program giving and algebraic-geometric machinery for classical mirror symmetry [GS03] The goal of this exposition is to outline the first steps of the Family-Floer program which shows that the SYZ correspondence can be used to produce homological mirrors.

1.1 SYZ: Some History

We start with some very geometric observations. Let X be a Calabi-Yau manifold, with holomorphic volume form $\Omega = \mathrm{Re}\Omega + i\mathrm{Im}\Omega$, and symplectic structure ω. These two forms equip X with 2 kinds of geometries (complex and symplectic) which we can study. Both complex geometry and symplectic geometry pick out preferred submanifolds of X, namely the *complex submanifolds* and *Lagrangian submanifolds* of X. At first glance, these geometries are surprisingly different: for instance, Lagrangian submanifolds are much more flexible than complex submanifolds; given any Lagrangian L, one can produce infinitely many different Lagrangians by taking Hamiltonian isotopies of L.

Due to the overabundance of Lagrangians, it makes sense to look at Lagrangians up to equivalence by Hamiltonian isotopy. In a Calabi-Yau manifold, we have a way to pick a preferred representative of a Hamiltonian isotopy class $[L]$ by picking the Lagrangian on which $\mathrm{Re}\Omega|_L = 0$. These are called the *special Lagrangian submanifolds* and they (at least locally) uniquely specify a Hamiltonian isotopy class of Lagrangians. A theorem of McLean [McL96] describes the deformation theory of special Lagrangians in terms of the $\mathcal{H}^1(L, \mathbb{R})$, the space of real harmonic 1-forms on L. This viewpoint of symplectic geometry draws many ideas from calibrated geometry, which concerns itself with volume minimizing submanifolds– here, the special Lagrangian is a volume minimizing representative of its Hamiltonian isotopy class.

The appearance of calibrated geometry, in retrospect, should be expected as a Calabi-Yau has 2 kinds of calibrated geometry: that coming from $\mathrm{Re}\Omega$, a non-vanishing closed n-form, and the geometry of ω^k, a non-vanishing closed $2k$ form. Where $\mathrm{Re}\Omega$ selects the calibrated special Lagrangian submanifolds, the calibrated geometry of ω^k exactly corresponds to the geometry of complex k-submanifolds. A particularly geometric interpretation of Mirror Symmetry posits that mirror spaces are those which interchange these two calibrated geometries. For instance, if X is a hyperkahler manifold of real dimension 4, the interchange between complex and symplectic geometry by hyperkahler twist exactly relates the special Lagrangian and complex submanifolds. This kind of intuition is true in the case of $K3$ surfaces, which are slef mirror (although, not simply by hyperkahler twist.)

This is where we begin our story with SYZ. Suppose that $F \subset X$ is a special Lagrangian torus. By the result of McLean, nearby special Lagrangian tori can be obtained identifying a neighborhood of F in X with T^*F, and looking at the section of a harmonic 1-form. As harmonic 1-forms are non-vanishing on the torus, each $q \in \mathcal{H}^1(F, \mathbb{R})$, gives us a new special lagrangian torus F_q disjoint from F. This gives us a map in a neighborhood of the origin

$$\mathcal{H}^1(F, \mathbb{R}) \times F \hookrightarrow X.$$

As the dimension $\dim H^1(F, \mathbb{R}) = n$, this is a diffeomorphism onto its image. We conclude that each Lagrangian torus gives us a small chart of a torus fibration on X.

The SYZ conjecture then posits that we can extend this local fibration to a global fibration of X by these Lagrangian tori. This assumption cannot be true on the nose for topological reasons [1] however, we could hope that it admits a fibration with some additional singular fibers.

Assumption 1.1.1. *There exists an almost toric special Lagrangian fibration of X,*

$$\begin{array}{ccc} F_q & \longrightarrow & X \\ & & \downarrow \\ & & Q \end{array}$$

called a SYZ*-fibration of X.*

For example, the product tori in a Toric Variety are Lagrangian, and are non-degenerate away from a codimension 2 manifold. One of the primary difficulties of extending SYZ mirror symmetry to is understanding how to incorporate degenerate tori into mirror symmetry constructions. For example, [Joy00] says that we should only expect to have a fibration away from a codimension 1 thickening of a codimension 2 critical locus.

If we take the existence of a SYZ fibration as a given, we can use the geometry of the fibration to construct a mirror for X. To each torus X we can associate a *dual torus* $\check{F}$ which is given by the moduli space of flat

[1] For instance, this would imply that $H_1(X) \neq 0$

$U(1)$ line bundles on T. This gives us a dual fibration over the same base, and we define $\check{X}$ to be the total space of this fibration,

$$\begin{array}{ccccc} F_q \longrightarrow X & & & & \check{X} \longleftarrow \check{F}_q \\ & \searrow & & \swarrow & \\ & & Q & & \end{array}.$$

This gives us the following conjecture / definition for mirror symmetry.

Conjecture 1.1.2. *The spaces X and $\check{X}$ are SYZ mirror.*

One approach to Mirror symmetry is now to understanding how SYZ-mirrors exhibit other predictions from mirror symmetry.

- McLean showed that there is a way interchange the complex structure on X with the symplectic structure on $\check{X}$ and vice-versa.
- The namesake of mirror symmetry is the interchange of Hodge numbers between X and its mirror space $\check{X}$. By a careful analysis of the Serre spectral sequence for computing homology (along with an adaptation to setting of almost fibrations,) Gross [Gro01] showed that if X and $\check{X}$ are 3-folds, we get the desired reflection of the Hodge diamond.
- By taking toric degenerations Gross and Seibert were able to relate open Gromov Witten invariants of X to period integrals on $\check{X}$.
- Perhaps most importantly, SYZ gives us a way to produce candidate mirror spaces to test our predictions on. For instance, mirror symmetry for Toric Fanos, hypersurfaces, and more have all been constructed using SYZ fibrations, and verified with computation.

The main goal of the Family Floer theory program is to show that SYZ mirrors are homological mirrors in the sense of [Kon94].

1.2 Homological Mirror Symmetry

Some Information about homological mirror symmetry should go here!

1.3 Outline of this exposition

The goal of this exposition is to outline some of the results and techniques of *family floer theory,* which provides a path to proving the following statement.

Conjecture 1.3.1. *Let $X, \check{X}$ be spaces which are SYZ mirror. Then X and $\check{X}$ are homologically mirror.*

In this paper, we will only sketch the existence of a functor

$$\mathcal{F} : \mathrm{Fuk}(X) \to Coh_{[\alpha]}(\check{X}),$$

and even this sketch will take a substantial amount of effort.

- In Section 2, we will do a second pass on the topology of SYZ fibrations, focusing on the affine structure on Q, and the presence of a $\mathcal{B}$-field gerbe that comes into play when trying to dualize the SYZ construction.
- In Section 3, we give a primer on rigid-analytic geometry. As end-users of this theory, we will only look at the shortcomings of using the näive topology on the mirror space $\check{X}$, and the route that rigid-analytic geometry takes to correct these difficulties. From this discussion, we'll see what kind of data we need to equip $\check{X}$ with in order to have a meaningful theory of coherent sheaves on it.

- In 4, we will use Floer theory to build local patches of sheaves on $\check{X}$, and build local patches of the family floer functor. We will also show that the Local patches of sheaves that we build are admissible with respect to the analytic structure on $\check{X}$.
- Finally, in 4.2, we show that our local patches glue together with the appropriate selection of gluing data.

2 A second look at SYZ topology

We will now take a more detailed examination of the geometry and topology that X inherits from an SYZ fibration. Since deformations of special Lagrangians are given by harmonic forms, we have an identification

$$T_qQ = H^1(F_q, \mathbb{R}), T^*_qQ = H_1(F_q, \mathbb{R}).$$

Inside these homology group there is a lattice of integral homology classes, which gives us a bundle of lattices

$$T^{\mathbb{Z}}Q \subset TQ, T^*_{\mathbb{Z}}Q \subset T^*Q.$$

These identifications give us for each F_q a preferred chart of Q, and since the transition maps between these charts will preserve integral cohomology classes, the space Q can be given an integral affine structure. The fiber can be recovered by taking the quotient

$$F_q = H^1(F_q, \mathbb{R})/H^1(F_q, \mathbb{Z}).$$

Note that while a choice of lattice $T^{\mathbb{Z}}Q \subset TQ$ induces an affine structure on Q, an affine structure does not recover this lattice (and therefore X cannot be recovered from only the affine base Q.) For example, $\check{X}$ and X are fibrations over the same affine base, but are not expected in general to be the same manifold.
The goal of this section is to motivate how the toric bundles X over affine Q can be specified by giving a Cech 2-cocyle in the sheaf of integral affine functions. This classification was originally understood using obstruction theory, and outlined in [GS03] We'll motivate the appearance of this class by appealing to mirror symmetry and the SYZ mirror construction, following [Hit01].

Reversing the SYZ Construction

As we've outline before, a Lagrangian torus fibration $F \to X \twoheadrightarrow Q$ gives us a dual torus fibration $\check{F} \to \check{X} \twoheadrightarrow Q$, where $\check{F}$ is the space of flat unitary connections on F. Using the bundles constructed before, one can also define the SYZ mirror as

$$\check{X} = T^{\mathbb{Z}}Q \otimes_{\mathbb{Z}} U(1).$$

If this a non-trivial bundle, there is no reason for us to expect the existence of a section (see the example below.) This means that $\check{X}$ has a very unusual torus fibration, as it admits a section $\check{s} : Q \hookrightarrow \check{X}$ by taking $\check{s}(q) \in \check{F}$ to be the trivial connection on F. [2] As a result, our SYZ mirror construction does not dualize in the sense that $\check{\check{X}}$ will *not* wind up being our original space X. This motivates the following question:

Question 2.0.1. *What is the additional data (X, B) and $(\check{X}, \check{B})$ that we need to make SYZ dual spaces mutually mirror to each other?*

[2] In light of homological mirror symmetry, we should completely have the expectation that $\check{X}$ admits a Lagrangian section, as the B-model on X has a structure sheaf which should be mirror to this section.

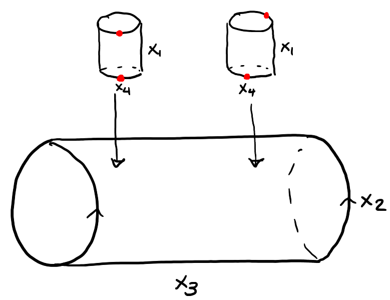
A Concrete Example: The Kodaira-Thurston Surface

To get a concrete example in mind, let's look at an example, due to Thurston [Thu76] with SYZ fibration considerations [AAOK]. Consider $\mathbb{R}^4$ with the standard symplectic form $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. We now look at two symplectomorphisms of the space $\mathbb{R}^4$

$$\begin{aligned} \phi : (x_1, x_2, x_3, x_4) &\mapsto (x_1 + 1, x_2, x_3, x_4 + x_3) \\ \psi_{ijk} : (x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2 + i, x_3 + j, x_4 + k) \end{aligned}$$

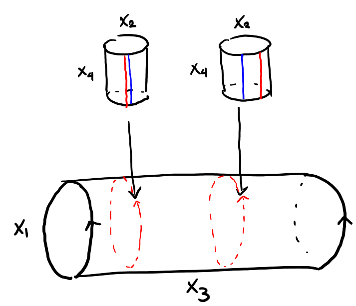
where $i, j, k \in \mathbb{Z}$. Let Γ be the group of symplectomorphisms generated by these two, and consider $X = \mathbb{R}^4/\Gamma$. There are two fibrations of $\mathbb{R}^4$ which descend to fibrations of X,

- The easier fibration involves projecting to coordinates x_2 and x_3. To visualize the construction: First quotient out by ψ_{ijk}. This is quotienting $\mathbb{R}^4$ by the subgroup $\{0\} \times \mathbb{Z}^3$, so the resulting quotient is $\mathbb{R}^4/\langle\psi_{ijk}\rangle = \mathbb{R} \times T^3$.



  We now quotient out by the action of ϕ, which acts on the cylinder $\mathbb{R}_{x_1} \times S^1_{x_4}$ by quotienting by $(x_1, x_4) \mapsto (x_1 + 1, x_4 + x_3)$ (So the cylinder is quotiented by a skew twist.) The base of this fibration is a torus, and admits a section given by $(0, x_2, x_3, 0)$.
  However, the dual-torus fibration to this is non-trivial, which is attributed to the monodromy of this fibration being non-trivial on homology (travelling in the x_3 direction corresponds to a Dehn twist on the fiber.) Even so, both X and $\check{X}$ have SYZ fibrations which admit Lagrangian sections. The Kodaira-Thurston surface has been proven to be *weakly self-mirror* in the sense of Merkolov. [Mer00].

- The more tricky fibration to view is given by projection to the x_1 and x_3 coordinates.
  The base of the fibration is a torus, given by $(x_1, x_3) \mapsto (x_1 + 1, x_3)$ and $(x_1, x_3) \mapsto (x_1, x_3 + 1)$.



  The torus fibers T^2 are then identified by $(x_2, x_4) \mapsto (x_2, x_4 + x_3)$. This give us a non-trivial fibration which does not admit a topological section. One can see the topology of this fibration in that going around the x_1 direction of the base gives me a rotation of x_3 in the x_4 component of the fiber. However, on the homology of the torus, this identification is trivial, so the bundle of local systems on these tori is trivial.

As a result, the mirror space constructed via this SYZ fibration is $\check{X} = T^2 \times T^2$. However, $T^2 \times T^2$ equipped with the standard SYZ fibration does not yield X again. The expectation is that X is not homologically mirror to $\check{X}$; instead, one needs to equip this space with additional data.

Problematically, a homological mirror symmetry construction using SYZ fibrations should predict that these two different mirrors have the same B-model. The work around to this is to expand our definition of B-model to include deformations by a $\mathcal{B}$-field, which will modify the B-model of $\check{X}$, and also provide us an additional geometric twist to the SYZ mirror construction.

Gerbes

The data of B comes in the form of a *gerbe.*[3] The full machinery of gerbes is probably too much to introduce in this talk. For us, a gerbe will determined by a S^1-valued Cech 3-cocycle[4]. Geometrically the information of a gerbe can be fit in a hierarchy with S^1-valued functions and unitary line bundles. Here is a chart on what geometric parallels we expect to exist for gerbes.

	Functions f	Line Bundles $\mathcal{L}$	Gerbes $\mathcal{G}$
Transition Data	$f_\alpha : U_\alpha \to S^1$	$g_{\alpha\beta} : U_\alpha \cap U_\beta \to S^1$	$h_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to S^1$
Compatibility	None!	$g_{\alpha\beta} = g_{\beta\alpha}^{-1}$	$h_{\alpha\beta\gamma} = g_{\beta\alpha\gamma}^{-1}$
Cocycle Condition	$f_\alpha f_\beta^{-1} = 1$	$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$	$h_{\alpha\beta\gamma} h_{\beta\gamma\delta}^{-1} h_{\gamma\delta\alpha} h_{\delta\alpha\beta}^{-1} = 1$
Difference of Trivializations		Functions	Line Bundles
"Characteristic Class"	$H^1(F, \mathbb{Z})$	$H^2(F, \mathbb{Z})$	$H^3(F, \mathbb{Z})$
Holonomy (if flat) along	Points	Curves	Surfaces

Notice that line bundles make two appearances in this hierarchy: either as geometric objects, or as the transition data of Gerbe trivializations. When line bundles are considered as geometric objects on their own, they form a group under tensor product, and come with a canonical identity object (the trivial line bundle). However, the set of trivializations of a Gerbe does *not* have a canonical identity, and forms a torsor. This means that once we fix a trivialization of a gerbe $\mathcal{G}$, the space of trivializations of $\mathcal{G}$ is identified with the space of line bundles; however, there is no reason to expect a canonical trivialization of a gerbe to exist. This gives us the following twisted SYZ construction.

Definition 2.0.2. *Let $\mathcal{B}$ be a flat Gerbe on X. Suppose that map $H^2(X, \mathbb{R}) \to H^2(F_q, \mathbb{R})$ is trivial. Then the restriction of $\mathcal{B}|_{F_q}$ is a flat gerbe with trivial holonomy. Let $\check{F}_{q,\mathcal{B}}$ be the dual torus of flat trivializations of $\mathcal{B}|_{F_q}$. We define the $\mathcal{B}$ twisted SYZ mirror to be the total space of the dual fibration given by $\check{F}_{q,\mathcal{B}} \to Q$*

Since the gerbe $\mathcal{B}$ is flat, we get a holonomy map from $C_2(X, U(1)) \to \mathbb{R}$, and therefore to each $\mathcal{B}$ we get an associated holonomy class $[b] \in H^2(X, U(1))$. Flat gerbes are determined by their holonomy, so specifying a class in $H^2(X, U(1))$ is the same a specifying a flat gerbe.

This class which naturally emerges in the gerbe story also makes an appearance from the topology of the SYZ fibration. Let P_i, P_j be two open sets, where we've built trivializations of $T^*_{\mathbb{Z}}Q$. The difference between these two trivializations is an exact differential form df_{ij}. Since the transitions are affine, this differential form will have integral coordinate values in each trivialization. Let $\mathcal{A}$ be the sheaf of functions with affine differential; then given trivializations of $T^*_{\mathbb{Z}}Q$ over P_i, P_j and P_k, we get a Cech 2-cycle

$$\alpha_{ijk} = f_{ij} - f_{jk} + f_{ik} \in \check{C}^2(X, \mathcal{A}).$$

[3]Gerbe (French): A spray, or sheaf

[4]Technical definition: A Gerbe is a stack of groupoids, which is *locally non-empty* and *transitive.* One way to relate this to our geometric intuition is that the sections over each open set consist of the category principle S^1 bundles, with morphisms given by isomorphism. In this language, it becomes clear the difference between two trivializations is a line bundle.
One can also fit this in our hierarchy, as a *line bundle* is a sheaf valued in the ring of functions.

This class $[\alpha]$, along with the affine structure of Q, specifies the fibration $X \to Q$ up to fiberwise symplectomorphism.[5]
Let $\check{X}$ be the standard SYZ mirror built from looking at the moduli of flat line bundles of F. Given an affine function $f : P \to \mathbb{R}$, we can associate a function $\exp(f) : \check{X}_P \to U(1)$ whose value at a point in the fiber (q, θ) is given by

$$\exp(f)(q, \theta) := \exp(i\langle df, \theta\rangle)$$

This exponential map extends to a map of homology groups

$$\exp : H^2(Q, \mathcal{A}) \to H^2(\check{X}, U(1)).$$

If we take the class $[\alpha]$ given to us by the SYZ fibration, our exponential map gives us a class $[\exp(\alpha_{ijk})] \in H^2(\check{X}, U(1)$ determining a flat gerbe $\mathcal{B}_\alpha$. So, the SYZ construction not only builds a mirror space $\check{X}$, but also canonically equips that mirror space with a flat gerbe. This constructed gerbe answers our question on how to dualize the SYZ construction.

Claim 2.0.3. *Let $\mathcal{B}_0$ be the trivial gerbe on X. Then $(X, \mathcal{B}_0)$ and $(\check{X}, \mathcal{B}_\alpha)$ are mutually twisted SYZ mirror to each other.*

Having understood the necessity of incorporating gerbes into discussions of SYZ fibrations, let's return to our concrete example, and look at implications of the presence of this gerbe to mirror symmetry predictions.

$\mathcal{B}$ -field

In general, Mirror symmetry is suppose to match the symplectic moduli of X with the complex moduli of $\check{X}$ and vice-versa. Problematically, the symplectic moduli of a space is a real space, while the complex moduli is a complex space. A solution to this is to incorporate the additional data of a B-field valued in $H^2(X, S^1)$, which complexifies the Kahler moduli space. The interpretation via gerbes gives a concrete meaning to the presence of this B-field, which tells us how which SYZ mirror we should take when performing homological mirror symmetry. The match between symplectic moduli of X and complex moduli of $\check{X}$ can be interpreted via deformations of categories,

$$H^2(X) \times H^2(X, S^1) \to \text{Deformations of } \omega \to HH^\bullet(\text{Fuk}(X)) \sim \text{Deformations of Fuk}(\check{X})$$

$$H^{1,1}(X) \to \text{Deformations of } J \to HH^\bullet(D^bCoh(X))$$

Here, the presence of a gerbe corresponds to a deformation of the symplectic category– however....

Twisted Sheaves

The last section of our discussion looks at how the data of this $\mathcal{B}$-field influences the A-model. Let's expand a little bit on this idea of deformation. By [HKR09],the Hochschild cohomology of $\check{X}$ giving the deformation theory of the DG category of coherent sheaves can be identified with polyvector fields on X, giving

$$HH^2(\check{X}) = H^0(\check{X}, \Lambda^2 T\check{X}) \oplus H^1(\check{X}, T_{\check{X}}) \oplus H^2(\check{X}, \mathcal{O}_{\check{X}})$$

These three components of homology can be understood as 3 different kinds of deformations

- $H^0(\check{X}, \Lambda^2 T\check{X})$ corresponds to non-commutative deformations.
- $H^1(X, T_{\check{X}})$ corresponds to deformations of complex structure. After the appropriate massaging, this is re-identified with $H^{1,1}(X)$, corresponding to the types of deformations that form the core of classical mirror symmetry.

[5]Notice, if $\alpha = 0$, then the space X is expected to be self-mirror. Examples include $X = (\mathbb{C}^*)^n$, or $X = T^{2n}$.

- $H^2(X, \mathcal{O}_{\check{X}})$ correspond to *gerby* deformations, where we loosen the cocycle condition on sheaves to the *twisted cocyle condition* that transition functions f_{ij} for a sheaf trivialization satisfies the condition

$$f_{ij} f_{jk} f_{ik} = g_{ijk} \, \mathrm{id}$$

instead. From this discussion, is evident that twisted sheaves form a DG category which is a possible receptacle for Homological Mirror Symmetry.

Lagrangian Sections

These two considerations directly relate to the existence (or non-existence) of a Lagrangian section of the fibration $F \hookrightarrow X \twoheadrightarrow Q$. The obstruction to constructing such a section turns out to give a codimension 2 obstruction in $H^2(Q, \mathcal{A})$, which is exactly this class α. The lack of a Lagrangian section means that we have no Lagrangian in $\mathrm{Fuk}(X)$ which represents the structure sheaf on $\check{X}$.
A work around to this is to only locally construct sections which are suppose to represent the structure sheaf; the failure of the cocycle condition to hold will end up factoring as additional data which we must record on $\check{X}$, in the form of a gerbe $\mathcal{B}_\alpha$.
In summary, we see the presence of gerbes in several components of our story. We started by introducing gerbes to balance out the SYZ construction, but this gerbe appears all over the place once we elevate it to an important position in the A and B model.

3 The Topology of the SYZ mirror

Homological mirror symmetry predicts a match between the Fukaya category of X and the derived category of coherent sheaves of the mirror space $\check{X}$. If we use the version of the SYZ construction above, our spaces X and $\check{X}$ will never be homologically mirror, as the Fukaya category is defined with Novikov coefficients. We therefore modify the SYZ construction to build from an Lagrangian torus fibration on X a *rigid analytic mirror* $\check{X}_\Lambda$.

3.1 Understanding SYZ with Novikov Coefficients

While the tori of an SYZ fibration have wonderful geometry, they are necessarily non-exact objects of the Fukaya category, and as a result any Floer theory that we attempt to define with these Lagrangians will necessarily be Novikov valued. Recall, the Novikov field is the set of formal sums with exponents increasing to infinity:

$$\Lambda := \left\{ \sum c_i T^{\lambda_i} \mid c_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim_{i \to \infty} \lambda_i = \infty \right\}$$

This field comes with a non-archimedian valuation

$$\begin{aligned} \mathrm{val} : \Lambda &\to \mathbb{R} \\ \sum_{\{\lambda_1 < \lambda_2 < \cdots \mid t_i \in \mathbb{R}\}} c_i T^{\lambda_i} &\mapsto \lambda_1 \end{aligned}$$

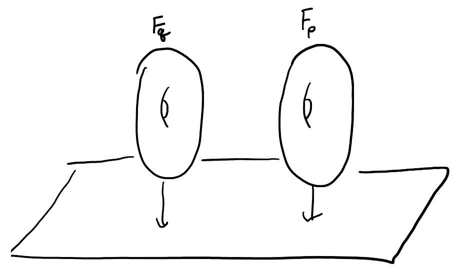
The *unitary* elements of Λ will be those with 0-valuation. The objects of the Fukaya category are Lagrangians L equipped with a Λ-unitary local system; by representing the monodromy of this local system with $b \in H^1(L, U_\Lambda)$, we can represent the objects of the Fukaya category as pairs (L, b)[6].
When we have an SYZ fibration, we now have a new candidate for what the mirror space should be: the moduli space of pairs (F_q, b)– as opposed to the moduli space of line bundles on F_q. By propegating this modification through the SYZ mirror construction, we build a SYZ mirror which is no-longer a complex

[6] A possible expansion of this story is to let b be the data of a bounding cochain on L. In the case where L bounds no holomorphic disks, this is exactly the data of a unitary local system.

manifold. Here is a dictionary between the SYZ construction with $\mathbb{C}$ coefficients, and the Λ-SYZ construction we will undertake:

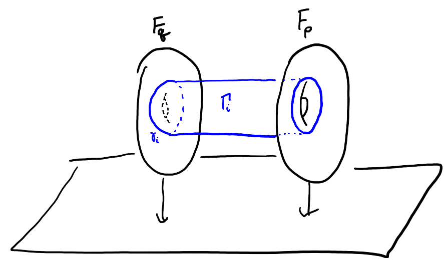
$\mathbb{C}$-SYZ	Λ-SYZ
$(F_q, \mathcal{L})$	(F_q, b)
$\check{F}_q = T_q^{\mathbb{Z}} Q \otimes U(1)$	$\check{F}_{q\Lambda} = T_q^{\mathbb{Z}} Q \otimes U_\Lambda$
$\check{X} := T^{\mathbb{Z}}\mathbb{Q} \otimes U$	$\check{X}_\Lambda := T^{\mathbb{Z}}\mathbb{Q} \otimes U_\Lambda$
$\log(\lvert \cdot \rvert) : \mathbb{C}^* \to \mathbb{R}$	$\mathrm{val} : \Lambda \to \mathbb{R}$
$\pi : \check{X} \to Q$	$\mathrm{val} : \check{X}_\Lambda \to Q$
$[\alpha] \in H^2(\check{X}, U(1))$	$[\alpha] \in H^2(\check{X}, U_\Lambda)$

Let's re-examine the local geometry of Lagrangian torus fibration.



The neighborhood of b_0 in B is canonically isomorphic to a neighborhood of) in $H^1(F_0, \mathbb{R}) \simeq \mathbb{R}^n$. We have that $F_b \subset$ Weinstein neighborhood of F, so that F_b corresponds to a graph of a closed 1-form on F_0, and we can take the associated cohomology class.
More explicitly, if we pick a basis γ_1, γ_n of $H_1(F_0)$, we can build cylinders Γ_i between the fibers corresponding to the trace of the γ_i. We can then define local coordinates by taking

$$p(b) := \left(\int_{\Gamma_1} \omega, \ldots, \omega \int_{\Gamma_n} \omega \right)$$



As we move around the torus fibration, we may have monodromy on $H_1(F_0)$ which gives us an integer affine structure on B.
Recall that $\Lambda = \{\sum a_i T^{\lambda_i} \mid \lambda_i \in \mathbb{R}, \lambda_i \to \infty\}$ has a valuation given by

$$\nu \left(\sum_i \Lambda_{a_i} T^{\lambda_i} \right) = \min\{\lambda_i \mid a_i \neq 0\} \in \mathbb{R}$$

and we have *unitary* elements given by $U_\Lambda = \nu^{-1}(0) \subset \Lambda^*$.
Given a chart for B near b_0, and given F_b, we get a rank 1 local system ξ on F_b with $\mathrm{hol}_\xi \in H^1(F_b, U_\Lambda)$. We want to associate to this a point in the candidate piece of mirror.

Associate to $z(F_b, \xi) \in H^1(F_{b_0}, \Lambda^*) \simeq (\Lambda^*)^n$ with $\nu(z) = p(b)$.
Concretely, fix $z = (z_1, \ldots, z_n) \in (\Lambda^*)^n$, we have that $z_i = T^{\omega(\Gamma_i)} \operatorname{hol}_\xi(\gamma_i)$. We now take

$$Y_P := \bigsqcup_b H^1(F_b, U_\Lambda) \simeq \nu^{-1}(P) \subset (\Lambda^*)^n.$$

Now this thing has an analytic structure. We now glue these together along the affine structure of B.

- Changing b_0 corresponds to rescaling the coordinates by T
- Changing the basis H_1 corresponds to a monomial transformation.

This viewpoint encourages thinking of F_p as something over F_q with a non-unitary local system.

3.2 The Topology

Our first step will be understanding how to equip $\check{X}_\Lambda$ with a topology so that the geometry of Novikov valuation and convergence is reflected in the map $\operatorname{val} : \check{X} \to Q$. After we have built this topology[7] we will have to understand what coherent sheaves are on this, and see how the class $[\alpha]$ from our previous discussion carries over to give us a rigid analytic gerbe on $\check{X}$. Ultimately, our intuition for the topology of $\check{X}$ will be informed by our expectations on how to compare the Floer homology of nearby fibers.
Before working on the topology of $\check{X}$, it is worth understanding how to give topology to some simpler spaces over Λ. Locally, one might expect the mirror $\check{X}$ to be constructed out of charts that look like the Novikov ring in k-variables. The difficulty of working with these spaces is that their standard topologies contain too many open sets, and therefore disconnected; secondly, the presence of so many open sets makes things that we would expect to be quasicompact are no longer compact. As a result, the classical topology on these spaces doesn't have a meaningful sheaf theory. Tate's solution to this problem is to judiciously remove open sets and covering maps until the topology was fine enough to do sheaf theory, but still coarse enough to not dissolve into a mess of disconnected sets.

The way that these open sets are selected is informed from trying to treat formal power series as the basis for covers in the same way that algebraic geometry uses polynomial rings for covers.

Definition 3.2.1. *Let Λ be a field with non-archimedean valuation* $\operatorname{val} : \Lambda^* \to \mathbb{R}$. *Give Λ the norm $|\cdot| : \Lambda \to \mathbb{R}$ given by $|f| = \exp(-\operatorname{val}(f))$.*
Define the valuation ring *to be the ring*

$$\Lambda_{\leq 1} = \{z \in \Lambda \mid |z| \leq 1\}$$

and the associated residue field $U_\Lambda = \Lambda_{\leq 1}/\mathfrak{m}$.

The key insight of rigid analytic geometry is that polynomial rings should not be the proper tool to build open charts in non-archmedian geometry, but rather small balls that look like the valuation ring.

Definition 3.2.2. *The n-variable* Tate algebra *over Λ is*

$$T_n := \{\sum f_A z^A \mid \operatorname{val}(f_A) \to \infty \text{ as } |A| \to \infty\}$$

The sup-*norm on T_n is*

$$\|\sum f_A z^A\| := \max_A |f_A| \geq 0.$$

The sup norm gives T_n topology of a Banach space, and it is multiplicative. The points of this chart are given by the maximal ideals of T_n, which we will denote $M(T_n)$[8].

[7] It will only be a Grothendeick Topology
[8] The M is for *Maximum Spec*

Remark 3.2.3. *In complex geometry, the maximal ideals of* $\mathbb{C}[x_i]$ *give us affine space. In rigid analytic geometry, the maximal ideals of the Tate algebra are points in* $\Lambda^n_{\leq 1}$. *Keep this in mind: we'll give some justification for this in a moment*

In classical algebraic geometry, we build spaces from gluing together from simple models, the *affine spaces* $\mathbb{C}[x_i]/I$. For Tate algebras, we can construct the basic building blocks of rigid-analytic geometry, the affinoids.

Definition 3.2.4. *A* affinoid algebra A *is a quotient of the Tate algebra* $A = T_n/I$.

Modding out by ideals of the form $(z-a)$ has a different effect than in complex geometry. For example, if we take $T_1/(z-f)$ we shouldn't expect the ring we get to be Λ, as the ideal $(z-f)$ need not be maximal. This is different then complex geometry, where $\mathbb{C}[x]/(x-a) = \mathbb{C}$.
There is map from $\Lambda \to T_1 \to T_1/(z-a)$, which is given by the inclusion to the constants. In the case of algebraic geometry, we would construct an inverse map by evaluating the polynomials at a; however, in our setting we cannot evaluate these polynomials at a, because there is no guarantee that the resulting power series would converge. The valuation of a^k may decrease faster than the coefficients of $\sum f_k z^k$ increase in valuation to infinity.
Let's look at exactly where our constructions fail to produce an isomorphism of ring. For this example, let's assume that the valuation of a is 1. Consider some power series

$$\sum f_k z^k$$

where the valuation of f_k is $k/2$, with vanishing constant term. If our intuition from polynomials were to hold, this would be some element of the ideal $(z-a)$. Our carryover proof would show that f_k is divisible by $(z-a)$ by greedily constructing a power series $\sum g_k z^k$ so that

$$(z-a)\sum g_k z^k = \sum f_k z^k.$$

We would start with

$$a \cdot g_1 = f_1$$
$$a \cdot g_2 + g_1 = f_2$$
$$a \cdot g_3 + g_2 = f_3$$

and so on. However, with this construction, the valuations of the g_i will be necessarily decreasing to $-\infty$, which is problematic!
On th plus side, this algorithm gives us a criteria for when points $T_1/(z-a)$ can be represented by elements of Λ. The only elements of $T_1/(z-a)$ which are contained in the image of Λ are those which have valuations satisfying the criteria

$$\left\{\sum f_k z^k \mid \lim_{i\to\infty} (\mathrm{val}(f_k) - k\,\mathrm{val}(a)) = \infty\right\}$$

which is one of these convergence criterias that we were hoping to find.
Notice that if the valuation of a is negative, this criteria is automatically satisfied, and $T_1/(z-a) = \Lambda$, which provides some intuition for why:

Claim 3.2.5. $M(T_1)$ *can be identified with* $a \in \Lambda_{\leq 1}$*, the "unit ball"* Λ.

Example 3.2.6. *A good example to look at is the charts for the mirror built in the SYZ construction. Let* P *be a polytope in* P *which is defined by looking by linear inequalites on* $H^1(F_q, \mathbb{R})$. *We define the affinoid chart around* q *to be given by*

$$\check{X}_P := \mathrm{val}^{-1}(P) = \bigsqcup_{p\in P} (H^1(F_p, U_\Lambda))$$

As a set of points, the chart of points in the mirror above this polytope is given by the max spec of what will eventually become the ring of functions on the mirror space $M(\mathcal{O}_p) = \check{X}_P$,

$$\mathcal{O}_P := \left\{ \sum f_A z_q^A \mid \forall v \in P, \lim_{|A| \to \infty} \mathrm{val}(f_A) + \langle v, A \rangle = \infty \right\}$$

This means for each $v \in A$, the minimal coefficients of f_A cannot decrease very rapidly. This corresponds to an intuition of convergence of these power series when evaluated on points in P. This example needs to be fleshed out in more detail *We can therefore express*

$$\mathcal{O}_P := T_n / I,$$

which shows that this basic building block of the SYZ mirror is an affinoid.

To say that $\mathcal{O}_P$ is the structure sheaf of the space $\check{X}_P$, we'll need to give $\check{X}_P$ a meaningful topology. One could simply equip $\check{X}$ with the Zariski topology, but this topology doesn't see the additional structure of the norm. The norm on Λ gives a canonical topology on $M(A)$. This isn't the topology that we'll end up working with, but it gives us a starting point for understanding the topology.
For this exposition, we'll make the simplifying assumption that that Λ is algebraically complete. To each point $x \in M(A)$, and each element $f \in A$, we get an element $f(x) \in \Lambda$ given by the quotient $\Lambda = A/x$. We therefore can assign a valuation to these pairs, and we define the open balls of radius $\epsilon > 0$ in $M(A)$ to be

$$B_{f,\epsilon} := \{x \in M \mid |f(x)| \leq \epsilon\}.$$

The canonical topology is defined as the topology on $M(A)$ generated from these open sets. Roughly each maximal ideal correspond to a point in the valuation ring $\Lambda^k_{\leq 1}$, [9] and we evaluate f at that point. The maximal ideals correspond to the zero loci. [10] The resulting output is a power series in λ, and we would like the leading order term of this evaluation to be sufficiently small– this means that the evaluation does not diverge.
Problematically, the topology we get from using all of these open sets is too fine. This is a problem inherited from the clopen basis that the nonarchimedean norm gives us [11]. Since this is a topology with clopen basis, the space is totally disconnected.
We're saved by taking a cue from algebraic geometry. In algebraic geometry, we have open sets coming from standard topology on Spec; however, one can also classify open affine subdomains via the universal properties they satisfy, and use this to define the open sets.

Definition 3.2.7. *A subset $U \subset M(A)$ is a* affinoid subdomain *if there is a morphism of affinoids* [12]

$$i : A \to A_U$$

so that

- *$i(M(A_U)) \subset U$ is contained in the affinoid subdomain.*
- *We have the following universal property: any other map of affinoids $\phi : A \to B$ whose map on MaxSpec lands in U factors through i.*

$$\begin{array}{ccc} M(B) & \xrightarrow{\phi} & U \subset M(A) \\ & \searrow & \uparrow i \\ & & M(A_U) \end{array}$$

[9]This is not obvious, but the rough argument: each maximal ideal gives us map $T_n \to T_n/\mathfrak{m}$, giving us an image of $z_i \in \Lambda$. The norm on Λ is multiplicative, but the norm on $T_n/\mathfrak{m}$ is only submultiplicative; by looking at the norms of $(z_i)^n$, one can conclude that since $|z_i| = 1$ the norm $|\phi(z_i)| \leq 1$, giving us an element of the valuation ring.

[10]In the case where Λ is not complete, not every point will be represented and we may have to work with extensions of Λ.

[11]A non-archimedean norm gives the space an ultrametric

[12]The morphisms between affinoids should be k-Banach algebras, so the norm is important here!

Open affine subschemes satisfy the same universal property; so one might make our definition of opens based on this affinoid property instead. The open balls that we constructed earlier are examples of affinoid subdomains, so this definition is compatible with our previous intuition of open. However, open affinoid subdomains do not enjoy all the same properties of open sets.

Claim 3.2.8. *Here are some properties of affinoid subdomains:*

- *Let $U, U' \subset M(A)$ be two affinoid subdomains. Then there intersection is an affinoid subdomian.*
- *If $\phi : M(B) \to M(A)$ is a map of affinoid spaces, then $\phi^{-1}(U)$ is an affinoid subdomian.*
- *$U' \subset U$ is an affinoid subdomian of $M(A)$ if and only if U' is an affinoid subdomain of $M(A_U)$.*

The most striking difference between affinoid subdomains and open sets is that there is no guarantee that the union of affinoid subdomains is again an affinoid subdomain. Fortunately, there is a machinery from algebraic geometry which can work with this initial data. Our substitute for a topology on $M(A)$ will be a category which has enough structure to perform sheaf theory (called a *Grothendieck topology.*

Definition 3.2.9. *A set $U \subset M(A)$ is a* admissible open *if it admits a covering by affinoid subdomains U_i so that for any map $\phi : M(B) \to M(A)$ with $\phi(M(B) \subset U)$, a finite subset of $\phi^{-1}(U_i)$ cover $M(B)$.*
We say that $\cup U_i = U$ is an admissible cover of admissible opens if whenever $\phi : M(B) \to M(A)$ has image $\phi(M(B) \subset U)$, the covering $\phi^{-1}(U_i)$ of $M(B)$ has a finite refinement by affinoids.
The Tate Topology *on $M(A)$ is the Grothendieck topology defined with objects admissible opens, and coverings given by admissible covers.*

To each admissible open, we have a sheaf $\mathcal{O}_U = A_U$. Tate's fundamental theorem shows that these assemble into a structure sheaf.

Theorem 3.2.10. *There us a unique extension of $\mathcal{O}_U = A_U$ to a sheaf $\mathcal{O}_A$. We have the left exact sequence*

$$0 \to \mathcal{A}_U \to \prod A_{U_i} \to \prod A_{U_i \cap U_j}.$$

This theorem allows us to build up rigid analytic spaces from gluing together affinoids, and define what coherent sheaves on rigid analytic space should be as well. Since we are allowed to build up rigid analytic spaces from affinoid covers, we can continue our process of building a mirror to X with the SYZ fibration. As previously discussed, the charts for the mirror built in the SYZ construction Y_P are (special) affinoid subdomains, and can be glued together to build a rigid analytic space, which now comes with a structure sheaf arising from the acylicity theorem.

Remark 3.2.11. *Technically, the transition maps that we have defined are only defined after making local choices of sections. The transition maps that we get do not satify the Cech cocycle condition, and so we do not get a structure sheaf on the mirror space $\check{X}$, but instead get a sheaf twisted by the gerbe $[\alpha]$ constucted before, which is a class that lives in $\prod A_{U_i \cap U_j \cap A_k}$.*

We conclude with a definition / outline of how to assemble the mirror space.

Definition 3.2.12. *Let $X \to Q$ be an SYZ fibration. The* rigid analytic SYZ mirror *is the space $\check{X} \to Q$. After picking a polyhedral decomposition on Q, the structure sheaf on $\check{X}$ is defined by aseembling the ring of functions be*

$$\mathcal{O}_P := \left\{ \sum f_A z_q^A \mid \forall v \in P, \lim_{|A| \to \infty} \operatorname{val}(f_A) + \langle v, A \rangle = \infty \right\}$$

on the affinoid subdomains $\check{X}_P$ into an analytic Gerbe on $\check{X}$ by using Tate's acylicity theorem.

4 Defining a Sheaf

Our definition of a sheaf will look a bit more combinatorial and algebraic than expected. This is overcome the following difficuties:

- First off, $\check{X}$ is not a topological space, so any definition of a sheaf that we have will have to involve a very combinatorial description based on the choice of Cech cover.
- Secondly, the way we constrcut a sheaf using family floer theory will mean we have to avoid some of the intutions of "sections over open sets," as when we pick an open set we will make choices of perturbation data to get a well defined floer theory. On intersections of these open sets, we may have made different choices of perturbation data, which will cause us to incorporate continuation maps of Floer homology.

Let's first describe the data we'll need to associate to a Lagrangian in order to combinatorially build a sheaf on the mirror. We will fix a cover of $\check{X}$ by polytopes which give us the affinoid subdomains that we used to construct the rigid analytic structure on $\check{X}$ earlier. We'll denote the set of vertices in this cover Σ, and we'll give Σ a order [13]. To each vertex $i \in \Sigma$, let P_i be the collection of all polytopes which contain i. We'll assume that our cover is fine enough that P_i again gives us a cover in our affinoid domain. This gives us a nice combinatorial description of our cover.
To ever subset $I \subset \Sigma$, let

$$P_I = \bigcap_{i \in I} P_i$$

Since the intersection of affinoids are again affinoids, this corresponds to some affinoid $\mathcal{O}_I$ representing the ring of functions over this open subset of the mirror.
The topology on $\check{X}$ can now be stated combinatorially by considering the category $\mathcal{O}_\Sigma$ be the category whose objects are ordered subsets $K \subset \Sigma$ and whose morphisms are defined by

$$\hom(I, J) = \begin{cases} \mathcal{O}_J & \text{If } I \subset J \\ 0 & \text{otherwise} \end{cases}$$

Remember that $I \subset J$ means that $P_J \subset P_I$, so the homomorphisms are restrictions: giving us simply multiplication by something on P_J.
A *presheaf* in this language is a functor

$$\mathcal{F} : \mathcal{O} \to \mathbf{Vect}(\Lambda).$$

For this to be a sheaf of $\mathcal{O}_\Sigma$-modules, we need compatibility with $\mathcal{O}_\Sigma$ so that the map

$$\mathcal{O}_J \otimes_{\mathcal{O}_I} \mathcal{F}(I) \to \mathcal{F}(J)$$

is an isomorphism.
To build the family Floer functor, to each lagrangian L we will assign create a Λ-vector space valued functor

$$J \mapsto \mathcal{L}(P_J).$$

We will construct this functor in 3 steps:

1. Construct a functor which assigns to each P_i a Λ-vector chain complex which is roughly given by the floer homology with a fiber above a point in that polytope,

   $$\mathcal{L}_i \sim CF(F_q, L).$$

   This assignment will not be functorial, because in the construction of the functor we will make choices dependent on the set i. We'll call this the *local mirror functor.*

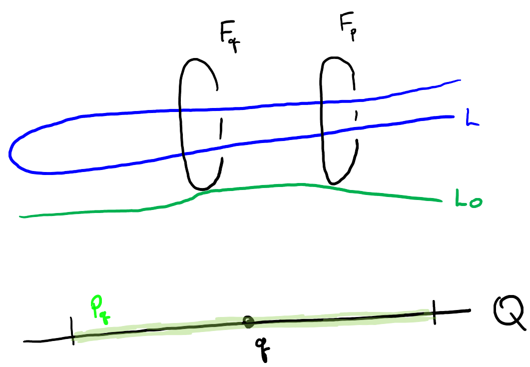
[13] This won't be important now, but becomes important later when we consider continuation data.

2. For a set J, we'll want to show that the choices made in the previous section lead to quasi-isomorphic chain complexes on the overlaps P_J. This will be the construction of $\mathcal{L}_J$ of continuation maps interpolating between the choices made earlier. The main difficulty here is constructing these continuation maps in such a way so that they themselves are compatible.

3. Show that the data that we've assembled in these quasi-isomorphisms make the homology of those chain complexes a $\mathcal{O}_\Sigma$-module sheaf via "homological patching."

4.1 Building the Chain Complexes

In this section, we construct the chain complexe $\mathcal{L}_i$ and argue why this is $\mathcal{O}_J$ module.



The zero section gives us a canonical point in each fiber. We have $CF((F,\xi, L) \simeq \bigoplus_{x\in F_b\mathcal{L}} \Lambda\langle x\rangle$. By equipping F_b with a base point $* = F_b \cap L_0$, we need to choose a homotopy class of path $*$ to x for each generator of the Floer complex. This gives us a trivialization of the local system ξ over each intersection point x.
We count holomorphic disks with weight $T^{\omega(u)}\,\mathrm{hol}(\partial U)$. Unless the portion of the boundary of boundary on F_b is a closed loop, there is no reason for these areas z_i to depend analytically on b. The trick is to do a change of basis. Let $g_x(p)$ be the area swept by the path $* \to x$ from the reference fiber F_{b_0} to F_b, which we will call Γ_x. We then set

$$\tilde{x} = T^{-g_x(p)}$$

. This means that we've rescaled the generators to take into account the possible wiggling of the chosen generator L_0.

Proposition 4.1.1 (Fukaya, Abouzaid)**.** *Given a disk u with a portion of boundary on F_b,*

$$\partial u \cap F_b = A \text{ path from } x \to y$$

Denote by $[\partial u] \in H_1(F_b)$ the class of the loop from $* \to x \xrightarrow{\partial u} y \leftarrow *$ Assume that u is a deformation of a similar disk u_0 with boundary of F_{b_0}. Then

$$T^{\omega(u)}\,\mathrm{hol}_\xi(\partial U) = T^{g_y(p)-g_x(p)}T^{\omega(u_0)}z(F_b,\xi)^{[\partial u]}.$$

The left hand side is the weight we are interested in counting. The right hand term is

- A term $T^{g_y(p)-g_x(p)}$ which will be removed via rescaling
- A constant $T^{\omega(u_0)}$ dependent on reference fiber
- Analytic Coordinates $z(F_b,\xi)^{[\partial u]}$.

Roughly, the contributions

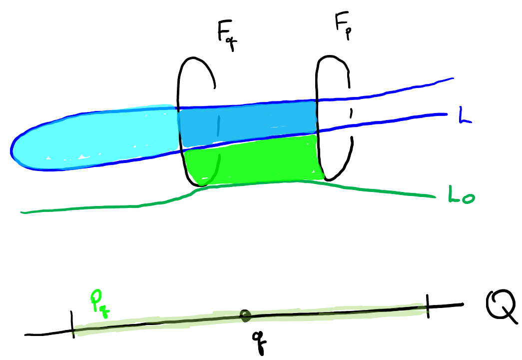
$$T^{\omega(u)}\,\mathrm{hol}_\xi(\partial U), T^{g_y(p)-g_x(p)}, T^{\omega(u_0)}$$

give us the cylinders which were used to construct our analytic coordinates in the first place.

$$\begin{aligned}[u] = [u_0] + \Gamma_y - \Gamma_x \\ =& [u_0] + \Gamma_y - \Gamma_x + \Gamma_{[\partial u]}\end{aligned}$$

so

$$\begin{aligned}\omega(u) = \omega(u_0) + g_y(p) - g_x(p) + \sum k_i p I9 \\ =& \omega(u_0) + g_y(p) - g_x(p) + \langle p(b), [\partial u]\rangle\end{aligned}$$



4.2 An examination of Continuation Data

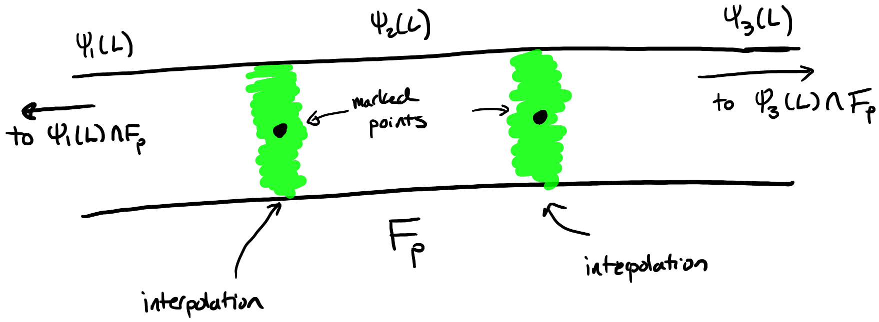
In the previous section, we chose data D_i to each vertex which gave us the required transversality. We'll want to show that the sheaf that we construct essentially does not depend on this choice of data. For this, we need to construct continuation maps. We now use the ordering of the vertices that we introduced before; we'll consider continuation maps which respect the ordering of the vertices.
Our goal: to associate to each subset I a continuation map

$$\mathcal{L}_I : \mathcal{L}_{\min I}|_{P_I} \to \mathcal{L}_{\max I}|_{P_I}[2 - |I|]$$

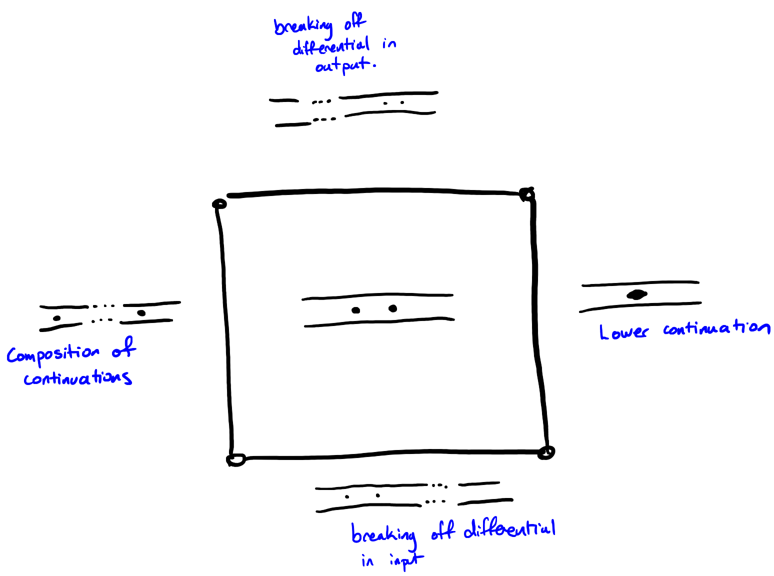
satisfying compatibility conditions. The easiest way to see that we need all of these maps is to look at the case where $I = \{i, j, k\}$. Then we have the following triangle of continuation maps

$$\begin{array}{ccc} \mathcal{L}_i & \xrightarrow{\mathcal{L}_{ik}} & \mathcal{L}_k \\ & {\scriptstyle \mathcal{L}_{ij}} \searrow \quad \nearrow {\scriptstyle \mathcal{L}_{jk}} & \\ & \mathcal{L}_j & \end{array}$$

which we can construct with traditional machinery from Lagrangian intersection Floer theory. However, we *additionally* need to know that there is compatibility between these three maps (so that this is a commutative diagram up to homotopy.) The data of $\mathcal{L}_{ijk}$ gives us the requisite compatibility condition; and L_I gives us higher homotopies that we'll need to show that the data we've chosen assembles into a sheaf. The moduli space giving the map L_I counts strips of the following form:



The marked points on the strips tell us how to interpolate between the various perturbation datum that we've chosen, so between the i and $i+1$ marked points we require the strip to satisfy the the floer equation for data associated to $\mathcal{L}_i + 1$. On the strip like ends, we ask these moduli spaces to satisfy the perturbation data for $\mathcal{L}_{\min I}$ and $\mathcal{L}_{\max I}$ respectively. The moduli space of such strips has boundary components that arise from 2 distinct phenomenon: it is possible that two of these marked points drift apart, leading to a breaking which corresponds to the composition of perturbation data, or it is possible that the points drift together, giving causing us to forget about some continuation map.



All in all, the map defined by counting elements of this moduli space satisfy the following relation:

$$m^1_{\max I}\mathcal{L}_I + \mathcal{L}_I m^1_{\min I} = \left(\sum_{\max I > i > \min I} \mathcal{L}_{i,\dots,\max I} \circ \mathcal{L}_{\min I,\dots,i}\right) + \left(\sum_{i<\max I} \mathcal{L}_{I\setminus i}\right)$$

A more detailed argument of the previous section confirms that these are maps of $\mathcal{O}_i$ modules.

4.3 Homological Patching

The last step is to take the above information and assemble it into a $[\alpha]$ twisted sheaf; for this, we need to show that the transition maps commute on the intersection up to a factor of α. To define this goes into more detail than we want to cover here, but essentially the same problem that we had before shows up here: is that there is not a consistant way to pick trivializations of the SYZ fibration over regions. As a result, the structure coefficients used to weight the strips in $\mathcal{L}_{ik}$ versus $\mathcal{L}_{jk} \circ \mathcal{L}_{ij}$ differ from eachother by the cocycle α.

References

[AAOK] Mohammed Abouzaid, Denis Auroux, Dmitri Orlov, and Ludmil Katzarkov. "Homological Mirror Symmetry for Kodaira-Thurston Manifolds". In: *In Preparation* ().

[Abo14a] Mohammed Abouzaid. "Family Floer cohomology and mirror symmetry". In: *arXiv preprint arXiv:1404.2659* (2014).

[Abo14b] Mohammed Abouzaid. "The family Floer functor is faithful". In: *arXiv preprint arXiv:1408.6794* (2014).

[Con08] Brian Conrad. "Several approaches to non-archimedean geometry". In: *p-adic geometry*. 2008, pp. 9–63.

[Gro01] Mark Gross. "Special lagrangian fibrations i: Topology". In: *AMS IP STUDIES IN ADVANCED MATHEMATICS* 23 (2001), pp. 65–94.

[GS03] Mark Gross and Bernd Siebert. "Affine manifolds, log structures, and mirror symmetry". In: *Turkish Journal of Mathematics* 27.1 (2003), pp. 33–60.

[Hit01] Nigel Hitchin. "Lectures on special Lagrangian submanifolds". In: *AMS IP Studies in Advanced Mathematics* 23 (2001), pp. 151–182.

[HKR09] Gerhard Hochschild, Bertram Kostant, and Alex Rosenberg. "Differential forms on regular affine algebras". In: *Collected Papers*. Springer, 2009, pp. 265–290.

[Joy00] Dominic Joyce. "Singularities of special Lagrangian fibrations and the SYZ Conjecture". In: *arXiv preprint math/0011179* (2000).

[Kon94] Maxim Kontsevich. "Homological Algebra of Mirror Symmetry". In: *in Proceedings of the International Congress of Mathematicians*. 1994.

[LYZ00] Naichung Conan Leung, Shing-Tung Yau, and Eric Zaslow. "From special lagrangian to hermitian-Yang-Mills via Fourier-Mukai transform". In: *arXiv preprint math/0005118* (2000).

[McL96] Robert C McLean. "Deformations of calibrated submanifolds". In: *Commun. Analy. Geom.* 1996.

[Mer00] SA Merkulov. "A note on extended complex manifolds". In: *arXiv preprint math/0005224* (2000).

[SYZ96] Andrew Strominger, Shing-Tung Yau, and Eric Zaslow. "Mirror symmetry is T-duality". In: *Nuclear Physics B* 479.1 (1996), pp. 243–259.

[Thu76] WP Thurston. "Shorter Notes: Some Simple Examples of Symplectic Manifolds". In: *Proceedings of the American Mathematical Society* 55.2 (1976), pp. 467–468.