An Exposition on the Chromatic Complex

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This exposition is designed to provide a short introduction to the "chromatic complex" of a graph. The paper is split into three parts.

1. The first part covers the definition of the chromatic complex
2. The second part looks at some operations that we can perform on graphs, and how we should think about these operations influencing the chromatic complex

3. In the third section, we find define a category where the operations of from the second part gives morphisms between graphs. We show that chromatic homology is a functor, answering a question presented in [5]

This paper draws from a lot of sources, but primarily Helme-Guizon's and Rong's paper [5] developing the chromatic complex, and Bar-Natan's outline of the Khovanov complex [2].

1 The Chromatic Complex

1.1 The Chromatic Polynomial

The chromatic polynomial was a graph invariant originally designed by George Birkoff [3] to attack the four color problem. The hope was that given a graph G with n vertices one could study the function $P_G(\lambda)$, which counted the number of different λ-colorings of a graph, with $\lambda < n$. More precisely, we can give a preliminary definition for the chromatic polynomial:

Definition 1.1 Preliminary

Let G be a graph on n vertices. Then a **λ-coloring** of G is an assignment of the numbers $1, 2, \dots, \lambda$ to the vertices of G so that adjacent vertices have different numbers. The **chromatic polynomial of** G (written $P_G(\lambda)$) is the unique degree n polynomial such that for all $0 \leq \lambda \leq n$, $P_G(\lambda)$ is the number of λ-colorings of G.

Example 1.2 Here is a quick computation of the chromatic polynomial for P_3, the three cycle. We notice that there are no 0-colorings, no 1-colorings, no 2-colorings, and 6 possible 3-colorings. The polynomial that fits this data is

$$P_{P_3}(\lambda) = \lambda(\lambda - 1)(\lambda - 2)$$

While outr preliminary definition gives one clear interpretation of what data is captured by the chromatic polynomial, it has some drawbacks. Firstly, in its current form, we would have to find all the colorings of a graph in order to compute the chromatic polynomial. The current definition doesn't show what topological aspects of the graph are being noticed by the polynomial. Finally, the current definition gives us a polynomial with coefficients in $\mathbb{Q}$, but we might be interested in working in a different ring. In order to remedy these shortcomings, we will develop a different definition of the chromatic polynomial that we will use from here on out.

Definition 1.3

Let G be a graph. Pick an edge $e \in E(G)$.

- The **contraction** of G by E (written G/e) is the graph where the vertices at the end of e have been identified.
- The **deletion** of e from G (written $G - e$) is the graph where the edge e has been removed from G.

Using these two operations, we can express the chromatic polynomial of a graph in terms of its contraction and deletions.

Claim 1.4 Let G be a graph, and pick an edge $e \in E$. Then

$$P_G(\lambda) = P_{G-e}(\lambda) - P_{G/e}(\lambda)$$

Proof. Let G be a graph, and let e be an edge of G connecting v and w.
If $v = w$, then $G - e = G/e$, and so we trivially have that $P_G(\lambda) = 0$, as expected.
So suppose that $v \neq w$. Then $P_G(\lambda)$ is the number of λ-colorings of $G - e$ where v and w do not share the same color. This is the total number of colorings of $G - e$, less the number of colorings where v and w have the same color, which is exactly $P_{G/e}(\lambda)$. ∎

Corollary 1.5 The coefficients of the chromatic polynomial lie in $\mathbb{Z}$.

From the last lemma, we can construct a definition of the chromatic polynomial that will be easier to compute, and can have coefficients in any ring that we would like.

Definition 1.6 Let G be a graph, and R a ring. Define the chromatic polynomial $P_{(G,R)}(\lambda) \in R[\lambda]$ to be the polynomial defined by the following relations

- If G is a collection of n points, then $P_{(G,R)}(\lambda) = \lambda^n$
- If e is an edge of G,
$$P_{(G,R)}(\lambda) = P_{(G-e,R)}(\lambda) - P_{(G/e,R)}(\lambda)$$

If we let $R = \mathbb{Z}$, we recover our original definition for the chromatic polynomial. But now we can look at the polynomial over a wider variety of rings, which could possibly give us more information on the graph. There is one last useful form of the chromatic polynomial, which gives us a bit of insight into what topological data is being stored by the polynomial.

Definition 1.7 Let $s \subset E(G)$ be a collection of edges. Let $[G : s]$ be the graph on the vertices of G, with edges s. Then we call $[G : s]$ the **resolution** of G by s. If G is a graph, then define $K(G)$ to be the **set of connected components** of G.

Here is a definition of the chromatic polynomial using resolutions of the graph.

Claim 1.8 Let G be a graph. Then the chromatic polynomial can be computed as
$$P_{(G,R)}(\lambda) = \sum_{s \subset E(G)} (-1)^{|s|} \lambda^{|K([G:s])|}$$
where $|K([G : s])|$ is the number of connected components of $[G : s]$.

Proof. For each edge e, let A_e denote the set of colorings of G by λ (not necessarily a λ-coloring) so that vertices of e are different colors. Then
$$P_G(\lambda) = \left| \bigcap_{e \in E} A_e \right|$$
Applying the principle of inclusion/exclusion gives the formulation above. ■

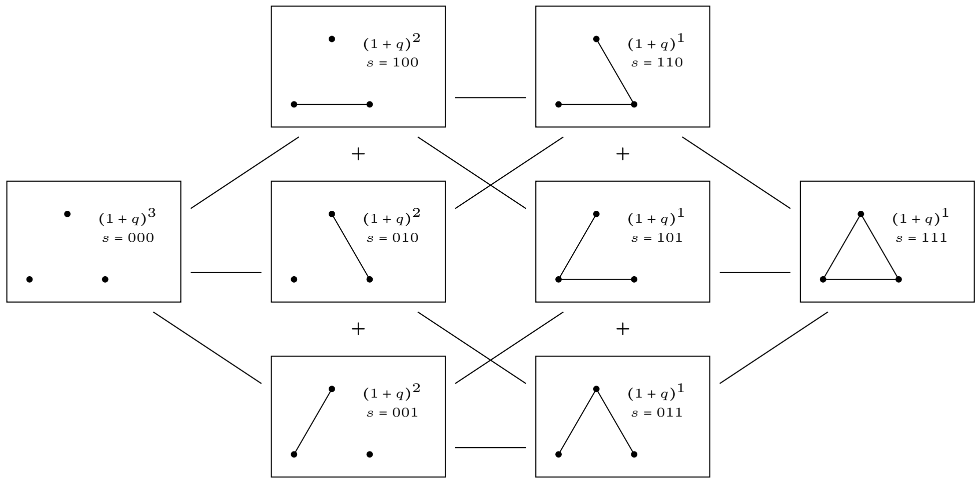
From here on out, it will be useful to rearrange the sum by the order of s, so we will frequently write
$$P_{(G,R)}(\lambda) = \sum_{i \geq 0} (-1)^i \sum_{\substack{s \subset E(G) \\ |s| = i}} \lambda^{|K([G:s])|}$$

Notice that this formulation again makes sense in any ring. A useful way of remembering this formula is with the following structure.

Definition 1.9 Let G be a graph. We will construct a diagram that computes the chromatic polynomial of G. Take the Hasse diagram for the boolean algebra on $E(G)$. For each subset $s \subset E(G)$, place the diagram of $[G : s]$ on the corresponding vertex of the Hasse diagram.

It is probably easiest to see this with an actual drawing of the cube of resolutions in Figure 1. Here, we will additionally label each vertex with $q^{|K([G:s])|}$, where $q = (\lambda - 1)$ and we compute $P_G(q + 1)$ by taking an alternating sum of the contributions of each diagram by column. Summing up the constribution from each resolution that we've drawn, we get $(1 + q)^3 - 3(1 + q)^2 + 3(1 + q) - q = (q - 1)q(q + 1)$ as expected. Notice here that we express the subset $s \subset E(G)$ as a string of 0 and 1's, so implicitly we have given the edge set

[h]



$(1+q)^3 \qquad -3(1+q)^2 \qquad +3(1+q) \qquad -1(q)$

Figure 1: The Cube of Resolutions for the graph P_3

$E(G)$ and ordering. We will later show that this ordering is not important, but useful to keep around for computations.

1.2 Quantum Dimension of Graded Modules

Our eventual goal is to construct a homology theory where the Euler characteristic of the chain complex is the chromatic polynomial. Before we construct to complex, we will first review some basic properties of graded dimension.

Definition 1.10 Let $M = \bigoplus_i M_i$ be a decomposition of a graded module M into its homogeneous submodules. Then the **graded dimension** (or sometimes **quantum dimension**) of M is the formal power series

$$\operatorname{qdim} M = \sum_i q^i \operatorname{rk}(M_i)$$

where $\operatorname{rk}(M_i) = \dim(M_i \otimes_R \operatorname{Frac}(R))$

Graded dimension plays very nicely with the operations of tensor product and direct sum. Let A and B be two graded modules. Then we have the two identities

$$\operatorname{qdim}(A \oplus B) = \operatorname{qdim}(A) + \operatorname{qdim}(B)$$

$$\operatorname{qdim}(A \otimes B) = \operatorname{qdim}(A) \cdot \operatorname{qdim}(B)$$

where $A \oplus B$ and $A \otimes B$ are given the natural gradings inherited from A and B.

Example 1.11 We give several examples of quantum grading that will be useful to remember in the future.

1. Let R be a ring. Then the ring $R[x]$ with the conventional grading has $\operatorname{qdim} R = 1 + q + q^2 + \ldots$.
2. Let $R[x]$ be as before. We define the module

$$V := R[x]/(x^2)$$

Then $\operatorname{qdim}(V) = 1 + q$. We will use this module over and over again.

3. Let V be as in the before example. Then

$$\operatorname{qdim}(V^{\otimes n}) = (1+q)^n$$

We are slowly building up a framework that allows us to do addition and multiplication of polynomials with algebraic spaces, instead of working with a ring. We finally will introduce one more idea.

Definition 1.12 A **graded chain complex** is a chain complex $C^\bullet$ where each C^i is a graded R module and the differentials d^i are graded maps. The homology groups $H^i = \frac{\ker d}{\operatorname{Im} d}$ inherit a grading from the chain complex and we define the **graded Euler characteristic** $\chi(C^\bullet)$ is the alternating sum of the quantum dimensions of the homology $C^\bullet$, that is

$$\chi(C^\bullet) = \sum_i (-1)^i \operatorname{qdim} H^i(C)$$

This definition is not entirely accurate— I've actually put up the definition of a graded cohomology theory. However, in the literature, it is almost always called a graded homology theory, and you could change the definition to be "proper" by inserting a few minus signs into the definition. At this point, we've developed the machinery necessary to categorify the chromatic polynomial.

1.3 Categorification of the Chromatic Polynomial

The idea behind the categorification is that we desire to replace the constants and variables that define the chromatic polynomial with R-modules. The chromatic complex $Ch^\bullet$ will have several advantages to the chromatic polynomial as an invariant associated to a graphs

- Since the chromatic complex has graded Euler characteristic of the chromatic polynomial, it contains at least as much data on the graph as the chromatic polynomial.
- Modifications of graphs by edge operations (addition, contraction, deletion) will correspond to chain maps on the chromatic complex
- We can use the tools of homological algebra to help us compute the chromatic complex.

With those thoughts in mind, let us define the groups in our chain complex.

Definition 1.13 Let G be a graph. Define the nth chain group of the **Chromatic Complex** to be the group

$$Ch^n(G) := \bigoplus_{\substack{s \subset E(G) \\ |s|=n}} V^{\otimes |K([G:s])|}$$

Where V is the group that we have defined earlier, $R[x]/x^2$ and $|K([G:s])|$ is the number of connected components of $[G:s]$. Notice that with the substitution $\lambda = 1+q$ we get

$$\operatorname{qdim} Ch^n(G) = \sum_{\substack{s \subset E(G) \\ |s|=i}} \lambda^{|K([G:s])|}$$

What we should think is that on top of every connected component of a resolution of the graph, we have stuck a copy of V. For this reason, it will be sometimes be convenient to think of the chain groups by $W_s = \bigotimes_{x \in K([G:s])} V_x$, where $x \in K([G:s])$ is a connected component of the s resolution. Then we can define the chain groups by

$$C^n(G) := \bigoplus_{\substack{s \subset E(G) \\ |s|=n}} W_s$$

At this point let's provide some motivation for why we are on the right track. If we were able to find a differential on this set of groups, would we get the right Euler characteristic. As $Ch^\bullet(G)$ is finite dimensional, we know that

$$\chi(Ch^\bullet(G)) = \sum_n (-1)^n \operatorname{qdim}(H^n(G)) = \sum_n (-1)^n \operatorname{qdim}(Ch^n(G)) = \sum_n (-1)^n \sum_{\substack{s\subset E(G)\\ |s|=i}} \lambda^{|K([G:s])|} = P_G(\lambda)$$

as desired. The second equality comes from the fact that the alternating sum of rank of homology is the same as the alternating sum of rank of complex. Now we just want to find a differential to make all of this work out.

Definition 1.14

We define the following multiplication, co-multiplication, unit and co-unit structure on the R-module V.

- The **multiplication** map, $m : V \otimes V \to V$ is defined as

$$m(1\otimes 1) = 1 \qquad m(1\otimes x) = x$$
$$m(x\otimes 1) = x \qquad m(x\otimes x) = 0$$

- The **comultiplication** map, $\Delta : V \to V \otimes V$ is defined as

$$\Delta(1) = 1\otimes 1 \qquad \Delta(x) = 1\otimes x + x\otimes 1$$

- The **unit** is the map $\epsilon : R \to V$ by $1 \mapsto 1$
- The **co-unit** is the map $i : V \to R$ by $1 \mapsto 0, x \mapsto 1$.

These maps give V the structure of a **Frobenius algebra**.

One thing that we should takeaway from this definition immedietly is that all of th emaps that we have defined above are graded maps. While we won't immedietly use the unit, co-unit and comultiplication structures in this section, we will need these properties later. We can now start to define the differential on the chain groups that we have set up for $Ch^\bullet(G)$.

Definition 1.15

Let $s < t$, (that is, $t\backslash s = \{e\}$, where e is some edge.) We want to define a map $d_{st} : W_s \to W_t$. We will first define component maps d_{st}^x where $x \in K([G:s])$ is a connected component. We break down into two cases dependent on x.

- Suppose that $x \in K([G:s])$ is a connected component that does not contain a vertex of the edge e. Then x can be matched to a connected component in $y \in K([G:s])$ which has the same edge and vertex set as x. We then define $d_{st}^x : V_x \to V_y$ by the identity.
- Suppose that $x \in K([G:s])$ is a connected component that does contain a vertex of the edge e. We break into two more cases
  - x is the only connected component that contains vertices from e. Then map $d_{st}^x : V_x \to V_y$ by the identity.
  - There is another connected component x' that contains a vertex from e. Then there is a connected component $y \in K([G:s])$ that is the union of x, x' and e. We define $d_{st}^{x,x'} : V_x \otimes V_{x'} \to V_y$ by the multiplication map m discussed earlier.

  We then define the **edge map** $d_{st} : W_s \to W_t$ as the tensor product of the component wise maps defined above.

The reason we call these maps edge maps is because we are going to place them on the edges of the giant cube of resolutions that we made earlier. At this point, the diagram gets a little messy. In Figure 2 I've added in a few of the edge maps so you can see where this complex is coming from.

Because the only non-identity portion of these edge maps came from a multiplication structure from an associative algebra, we know that (if we left out the $\oplus$ signs) this is a commutative diagram. We now need to find a way to "compress" our edge morphisms $d_{st} : W_s \to W_t$ into a chain map from $C^n \to C^{n+1}$. The

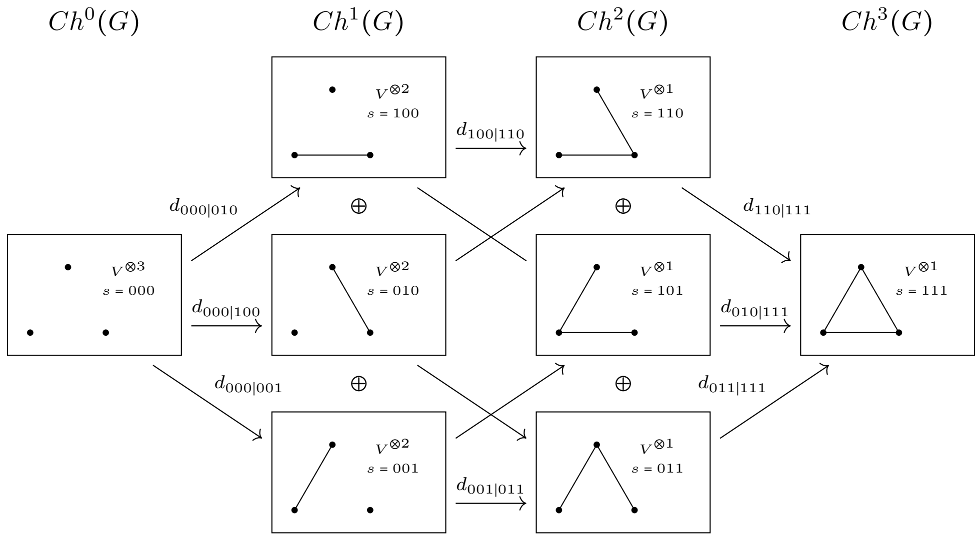


Figure 2: The cube of resolutions for P_3, with added morphisms and labels.

way we do this is in a manner very similar to construction of the total differential on a bicomplex. We will "sprinkle" minus signs among the edge maps so that every square in the diagram above anticommutes, and then we will let the boundary map on the complex be the sum of these signed differentials. To figure out where the minus signs should go, recall that we have assigned an ordering to $E(G)$ which makes every subset s expressible as a binary string. This ordering will help us determine where to place these minus signs; later we will show that our homology theory is independent of that ordering.

Definition 1.16 The differential on the chromatic complex $d^i : Ch^i \to Ch^{i+1}$ is defined by the sum

$$d^i = \sum_{\substack{|s|=i \\ s<t}} (-1)^{\sigma(st)} d_{st}$$

Where $\sigma(st)$ is the first place that s and t differ given some fixed ordering of $E(G)$.

As the multiplication map is graded, the differential is a graded map on the $Ch^i(G)$. The anticommutivity of the cube of resolutions and work above gives us

Theorem 1.17 The map above has the property $d^2 = 0$, and $Ch^\bullet(G)$ is a chain complex with graded Euler characteristic the chromatic polynomial.

1.3.1 Enhanced States

While the description above provides intuition for where the chromatic complex comes from, it is sometimes easier to work with a description of the chromatic complex that explicitly writes out the basis for the theory. This description is called "enhanced states", which was presented in [5] and based on [12] description of Khovanov homology.

Definition 1.18 Let G be a graph. An **enhanced state** of G is a pair (s, c), where $s \subset E(G)$ and c is an assignment of 1 or x to each connected component of $[G : s]$.

This is easily seen to be a basis of the of the complex that we had described earlier, with $W_s = \bigoplus_c (s,c)$. Define $|s|$ to be the number of edges in s, and $|c|$ to be the number of x present in the coloring. Then $|s|$ is the *homological index* of the basis element (s,c), and $|c|$ is the *quantum grading* of the basis element (s,c). We just need to define where the differential sends each on of these enhanced states.
Define the differential by $d(s,c) = \sum_{s<t}(-1)^{\sigma(st)}(t,c_e)$ where $\sigma(st)$ is the edge sign assignment function from the earlier section, and c_e is a coloring of t that arises by

- If a connected component of t does not contain the edge e, it is colored with the same color it had in (s,c).
- If 1 connected component of s contains the edge e, then the connected component of t containing the edge e is colored the same way.
- If 2 connected components of s contain the edge e, then the connected component of t containing the edge e is colored with the product of those two colorings via the multiplication map defined earlier.

One chan check that this definition of the differential is exactly the same as the one defined above.

1.4 An Example Computation

In this section, we explicitly compute the chromatic homology for P_3, the three cycle working with coefficients in $\mathbb{Z}$. It will be most convenient to represent the generators of the chain complex in a grid, with one axis being the complex (or sometimes called homological) grading, and the other axis representing the quantum grading. The generators for the chain complex of $Ch^\bullet(G)$ are written out on this table.

	$Ch^0(G)$	$Ch^1(G)$	$Ch^2(G)$	$Ch^3(G)$
3	$x\otimes x\otimes x$			
2	$1\otimes x\otimes x$ $x\otimes 1\otimes x$ $x\otimes x\otimes 1$	$(x\otimes x,0,0)$ $(0,x\otimes x,0)$ $(0,0,x\otimes x)$		
1	$1\otimes 1\otimes x$ $x\otimes 1\otimes 1$ $1\otimes x\otimes 1$	$(1\otimes x,0,0)$ $(0,1\otimes x,0)$ $(0,0,1\otimes x)$ $(x\otimes 1,0,0)$ $(0,x\otimes 1,0)$ $(0,0,x\otimes 1)$	$(x,0,0)$ $(0,x,0)$ $(0,0,x)$	x
0	$1\otimes 1\otimes 1$	$(1\otimes 1,0,0)$ $(0,1\otimes 1,0)$ $(0,0,1\otimes 1)$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$	1

Now that we have the generators picked out, a computation shows that these elements listed in the table below generate the kernel of $d^\bullet$.

	$Z^0(G)$	$Z^1(G)$	$Z^2(G)$	$Z^3(G)$
3	$x\otimes x\otimes x$			
2	0	$(x\otimes x,0,0)$ $(0,x\otimes x,0)$ $(0,0,x\otimes x)$		
1	0	$(1\otimes x - x\otimes 1,0,0)$ $(0,1\otimes x - x\otimes 1,0)$ $(0,0,1\otimes x - x\otimes 1)$ $(1\otimes x,-1\otimes x,1\otimes x)$	$(x,x,0)$ $(x,0,-x)$	x
0	0	$(1\otimes 1,-1\otimes 1,1\otimes 1)$	$(1,1,0)$ $(1,0,-1)$	1

If we compute the images , we get the following images.

	$C^0(G)$	$C^1(G)$	$C^2(G)$	$C^3(G)$
3	0			
2	0	$2(x\otimes x,x\otimes x,x\otimes x)$		
1	0	$(x\otimes 1,-1\otimes x,x\otimes 1)$ $(1\otimes x,-x\otimes 1,x\otimes 1)$ $(x\otimes 1,-x\otimes 1,1\otimes x)$	$(x,x,0)$ $(x,0,-x)$	x
0	0	$(1\otimes 1,-1\otimes 1,1\otimes 1)$	$(1,1,0)$ $(1,0,-1)$	1

Computing the homology yields

	$H^0(G)$	$H^1(G)$	$H^2(G)$	$H^3(G)$
3	[x, x, x]			
2	0	$[(x \otimes x, x \otimes x, x \otimes x)] \mod 2$		
1	0	$[(x \otimes 1, -x \otimes 1, 1 \otimes x)]$	0	0
0	0	0	0	0

Notice that if instead we work over different coefficients, we get slightly different homology groups.

	$H^0_{\mathbb{Z}}$	$H^1_{\mathbb{Z}}$	$H^2_{\mathbb{Z}}$	$H^3_{\mathbb{Z}}$
3	$\mathbb{Z}$			
2		$\mathbb{Z}_2$		
1		$\mathbb{Z}$		
0				

	$H^0_{\mathbb{Q}}$	$H^1_{\mathbb{Q}}$	$H^2_{\mathbb{Q}}$	$H^3_{\mathbb{Q}}$
3	$\mathbb{Q}$			
2				
1		$\mathbb{Q}$		
0				

	$H^0_{\mathbb{F}_2}$	$H^1_{\mathbb{F}_2}$	$H^2_{\mathbb{F}_2}$	$H^3_{\mathbb{F}_2}$
3	$\mathbb{F}_2$			
2	$\mathbb{F}_2$	$\mathbb{F}_2$		
1		$\mathbb{F}_2$		
0				

There are good reasons to work with coefficients not in $\mathbb{Q}$. For example, it is known that the chromatic polynomial completely determines the chromatic homology with coefficients in $\mathbb{Q}$ [4], but the torsion in other theories remains interesting.

2 Operations on Graphs

One of the promises made in the beginning of this paper was that operations between graphs would induce a map on homology of a graph. The following four operations on graphs were developed in [5].

- Graph Isomorphism
- Edge Removal (Inclusion)
- Graph Expansion (Projection)
- Disjoint Unions

We will develop additional operations, which are based on analogous constructions in Khovanov homology exhibited in [1].

- Merging (Multiplication)
- Splitting (Comultiplication)
- Adding a point (Unit)
- Removing a point (Co-unit)

In this section we will be treating operations as if they were "morphisms" between graphs, but it won't be until Section 3.1 that we'll be able to make this structure formal. Finally, we use some of these mappings to construct a exact sequence of the chromatic cohomology.

2.0.1 Graph Isomorphism

Our goal here is to show that the chromatic complex is an invariant of the isomorphism type of a graph, as opposed to just the graph with a prescribed ordering of the edges.

Definition 2.1 Let G, H be graphs. A **graph isomorphism** is a bijection $f : V(G) \to V(H)$ such that respects the edge relations of G and H.

Claim 2.2 A graph isomorphism $f : G \to H$ induces a chain isomorphism $f : Ch^\bullet(G) \to Ch^\bullet(H)$.

Proof. It is easiest to prove this using exact states. A graph isomorphism is nothing more than a reordering of the vertices between graphs, so it suffices to show that for any transposition of labelings you get the same chain complex. We prove this following the outline in [5]

Let $p = (k, k+1)$ be a transposition of two different labels in the vertex set, and let G and G_p be two graphs whose vertex set differ by this labeling. Define the map $f : Ch^\bullet(G) \to Ch^\bullet(G_p)$ by taking the enhanced states $(s,c) \mapsto (-1)^{\tau(p,s)}(p(s), p(c))$ where $\tau(p,s) = 1$ if both k and $k+1$ show up in the set s, and zero otherwise. This is clearly an isomorphism of R modules, so we just need to show that this is a chain map. This is just a computation to show that the sign change added by $\tau(p,s)$ exactly compensates for the sign change caused by reordering of the edge sets. ∎

This shows that the chromatic complex is actual an invariant of the isomorphism type of a graph, and that the ordering of the labelings that we've been using this whole entire time to describe the chain complex is not so important.

2.0.2 Graph Expansion

Definition 2.3 Let G be and H be two graphs. Suppose that there is a subgraph $K \subset H$ such that $G = H/K$. Then we say that H is an **expansion** of G by K, and we write $i_K : G \to H$ to represent this expansion.

Claim 2.4 Suppose that $i_K : G \to H$ is an expansion of a graph. Then we get an induced inclusion of chain complexes $i_K : Ch^\bullet(G) \to Ch^\bullet(H)$

Proof. We may assume that K consists of a single edge, and iterate this process to handle the case where K is a subgraph consisting of several edges.
As $Ch^\bullet(G)$ is an invariant up to isomorphism type, we may assume that the order of the labelings of edges for H and G are such that the e is the last label of H and the $E(G) = E(H)\backslash\{e\}$ as ordered sets. Let (s,c) be an enhanced state of G. Then $s \cup \{e\}$ is a subset of $E(H)$, and c defines a coloring on $s \cup \{e\}$ as expansion does not change the number of connected components. Define $i_e((s,c)) = (s \cup \{e\}, c)$. Then one can check that this is an inclusion of $Ch^\bullet(G)$ into $Ch^\bullet(H)$. Because of our choice of ordering of edges, all of our resolutions will match up, and so the map commutes with the differential. ∎

A natural way to see this map is that all the states of $Ch^\bullet(H)$ that contain e form a subcomplex of $Ch^\bullet(H)$ that can naturally be identified with $Ch^\bullet(H/e)$. This is probably most easily seen via a diagram (Figure 3)

2.0.3 Edge Removal

Definition 2.5 Let G be and H be two graphs. Suppose that there is a subgraph $K \subset H$ such that $G = H\backslash E(K)$. We call this an **edge removal** and write the map $\pi_K : H \to G$ to represent this edge removal.

Claim 2.6 Suppose that $\pi_k : H \to G$ is an edge removal. Then this induces a morphism $\pi_K : Ch^\bullet(H) \to Ch^\bullet(G)$.

Proof. Again, we do this using enhanced states. We may arrange order the edges of H so that the edge that is removed is the last edge in any ordering. Then we define

$$\pi_e((s,c)) = \begin{matrix} (s,c) & \text{if } e \text{ is not in } s \\ 0 & \text{otherwise} \end{matrix}$$

One can check that this is a morphism of chain complexes, but it is probably easiest seen by just looking at Figure 4. ∎

2.0.4 Disjoint Unions

The chromatic complex for the disjoint union follows that pattern that would be expected from the Künneth formula.

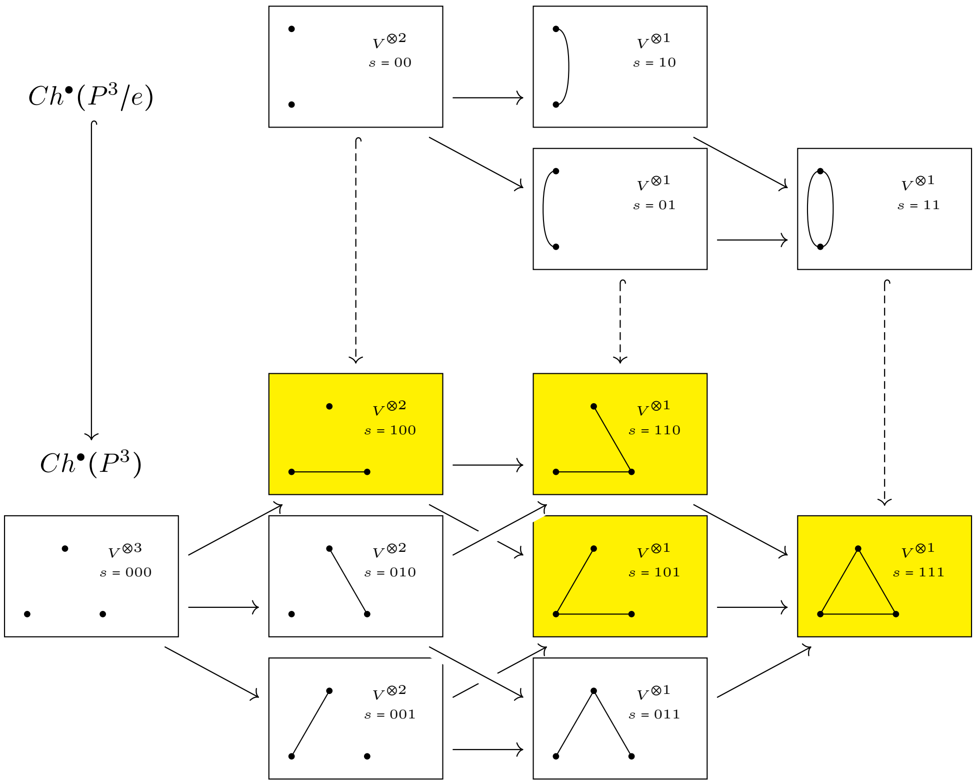


Figure 3: A diagram for the graph expansion morphism. The complex on the top represent $Ch^\bullet(G/e)$, while the complex on the bottom is $Ch^\bullet(G)$. The highlighted region shows where the image of the inclusion.

Claim 2.7 Let G, H be graphs. Then $Ch^\bullet(G_1 \sqcup G_2) = Ch^\bullet(G_1) \otimes Ch^\bullet(G_2)$. The homology is given by the Künneth formula,

$$H^i(G \sqcup H) = \left(\bigoplus_{p+q=i} H^p(G) \otimes H^q(H) \right) \oplus \left(\bigoplus_{p+q=i+1} H^p(G) * H^q(H) \right)$$

where $*$ is the torsion product of abelian groups.

2.0.5 Multiplication/Merging

In addition to the inclusion and projection formulas that we have above, we could look at different operations of graphs that use the multiplication and comultiplication structures that we developed earlier. The first one that I would like to look at is multiplication. When we defined the complex, we used multiplication to define how states should act when we add an edge between them. We can extend this operation on just the states to the whole entire graph.
The corresponding operation on the graph is called "merging."

Definition 2.8 Let H be a graph, and $v_1, v_2 \in V(H)$.,Let $H \cdot (v_1 v_2)$ be the graph which is obtained by taking H removing vertices v_1 and v_2, and adding in a vertex v with the edge set of v_1 and v_2. We call $H \cdot (v_1 v_2)$ the **merging** of H along v_1 and v_2 and we write this as $m_{v_1 v_2} : H \to H \cdot (v_1 v_2)$.

Claim 2.9 If $m_{v_1 v_2}$ is a merging of H to $H \cdot (v_1 v_2)$ along v_1 and v_2, there is an induced map $m_{v_1 v_2} : Ch^\bullet(H) \to Ch^\bullet(H \cdot (v_1 v_2))$.

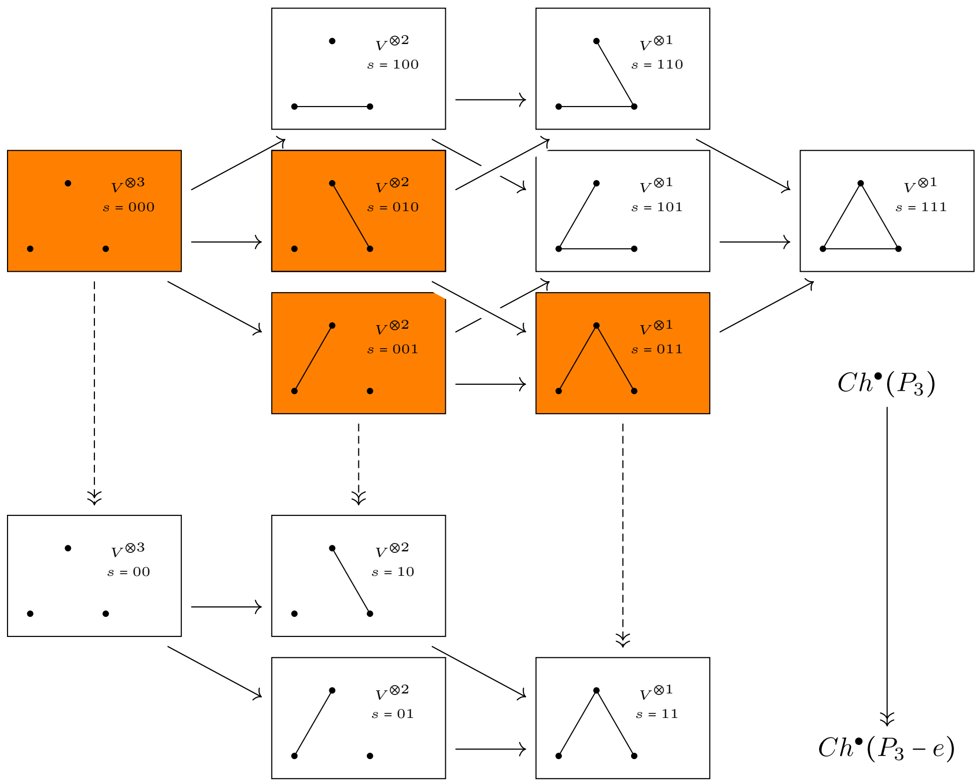


Figure 4: Diagrams representing the edge removal map. The upper complex is $Ch^\bullet(G)$ and the lower complex is $Ch^\bullet(G-e)$. The highlighted region represents the identification of $Ch^\bullet(G-e)$ with a subspace of $Ch^\bullet(G)$ as vector spaces.

Proof. We just need to exhibit what this map is on enhanced states. Notice that both $H\cdot(v_1v_2)$ and H have the same edges, and their vertex sets differ by a spot where two vertices v_1, v_2 have been "glued" together to a new vertex v. So let us assume that the ordering of edges in $H\cdot(v_1v_2)$ and H are the same. While $H\cdot(v_1v_2)$ and H have the same edge labelings, they do not have the same enhanced states. This is due to the fact that each resolution $[H\cdot(v_1v_2):s]$ may have one fewer connected component than $[H:s]$. However, each connected component of $[H\cdot(v_1v_2):s]$ is naturally identified to either 1 or 2 connected components of $[H:s]$, and so it makes sense to define $m_{v_1v_2}(s,c) = (s,c')$ where c' is the following coloring

- If a connected component of $[H\cdot(v_1v_2):s]$ does not contain v, then it gets the same coloring as the one it is associated to in $[H:s]$.
- If a connected component of $[H\cdot(v_1v_2):s]$ contains v, then look at v_1 and v_2 in $[H:s]$.
  - If v_1 and v_2 contained in the same component of $[H:s]$, then give the component containing v in $[H\cdot(v_1v_2):s]$ that coloring
  - If v_1 and v_2 are contained in two different components of $[H:s]$, then give the component containing v in $[H\cdot(v_1v_2):s]$ the coloring corresponding to the product of those two components' colors.

Again, this is probably easiest seen by the diagram in Figure 5 , so we will include one.

The maps in red represent the map m. ■

2.0.6 Comultiplication/ Splitting

Similarly, if we take a graph and split a vertex into two vertices, we would like to get a map between the chain complexes. As splitting should be somehow opposite to merging, we will use the comultiplication

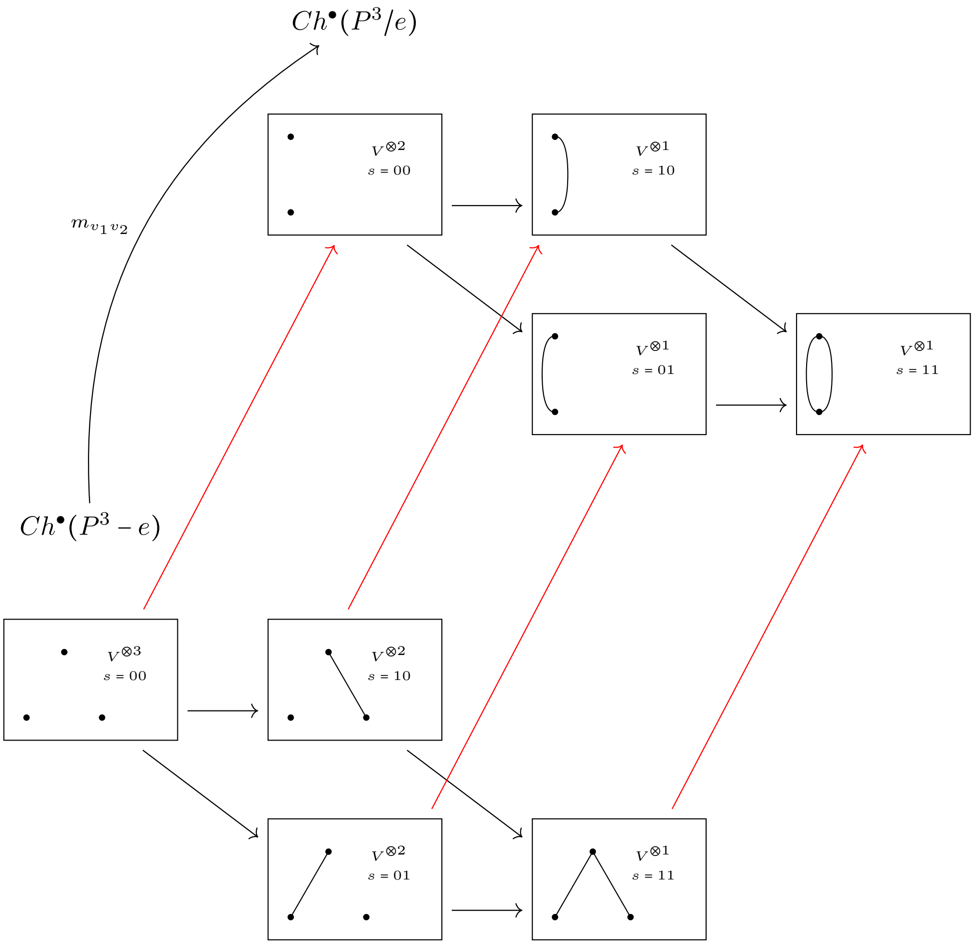


Figure 5: A multiplication map between two complexes. Each highlighted arrow represents one multiplication map of connected components.

structure on our Frobenius algebra V to define this map.

Definition 2.10 Let G be a graph. Pick a vertex v of G, and two sets E_1 and E_2 which partion the edge incident to v. Let $G \div (E_1E_2)$ be the set obtained by removing v, adding in two vertices v_1 and v_2, and letting the edge set of v_1 by E_1 and the edge set of v_2 be E_2. We call $G \div (E_1E_2)$ the **splitting** of G by E_1 and E_2 at v, and we write this as $\Delta_{E_1E_2} : G \to G \div (E_1E_2)$.

Splitting and merging are opposite operations in the sense that $(G \div (E_1E_2)) \cdot (v_1v_2) = G$ and $(G \cdot (v_1v_2)) \div (E_1E_2) = G$.

Claim 2.11 Suppose $\Delta_{E_1E_2} G \to G \div (E_1E_2)$ is a splitting along E_1 and E_2. Then this induces a map $\Delta_{E_1E_2} : Ch^\bullet(G) \to Ch^\bullet(G \div (E_1E_2))$.

Proof. We will exhibit a map on the enhanced states, and then show that this map commutes with the differential. As with multiplication, we know that G and $G \div (E_1E_2)$ have the same edge sets, so let us order them the same way. The only difference between an enhanced state of G and one of $G \div (E_1E_2)$ is that $G \div (E_1E_2)$ may have one more connected component per resolution than G does. Let (s, c) be an enhanced state of G. Then define $\Delta_{E_1E_2}(s, c)$ as follows.

- Suppose v_1, v_2 are contained in the same component of $[G \div (E_1E_2) : s]$. Then every connected component of $[G \div (E_1E_2) : s]$ corresponds to one connected component of $[G : s]$, so we can give them the same coloring.
- Otherwise, v_1 and v_2 do not belong to the same connected component of $[G \div (E_1E_2) : s]$.
  - If the connected component that contains v is colored 1, then $\Delta_{E_1E_2}(s,c) = (s,c')$ where every component of $[G \div (E_1E_2) : s]$ is colored with the component from which it comes from– so the component containing v_1 is colored 1, and the component containing v_2 is colored 1.
  - If the connected component that contains v is colored x, then $\Delta_{E_1E_2}(s,c) = (s,c_1) + (s,c_2)$, where c_1 is the coloring where the component containing v_1 is colored x while the component containing v_2 is colored 1. Likewise c_2 is the coloring where the component containing v_2 is colored x and v_1 is colored 1.

This map naturally comes from the comultiplication structure that we defined earlier. To get a handle on the map, here is another diagram (Figure 6). Why does this map commute with the differential? This is

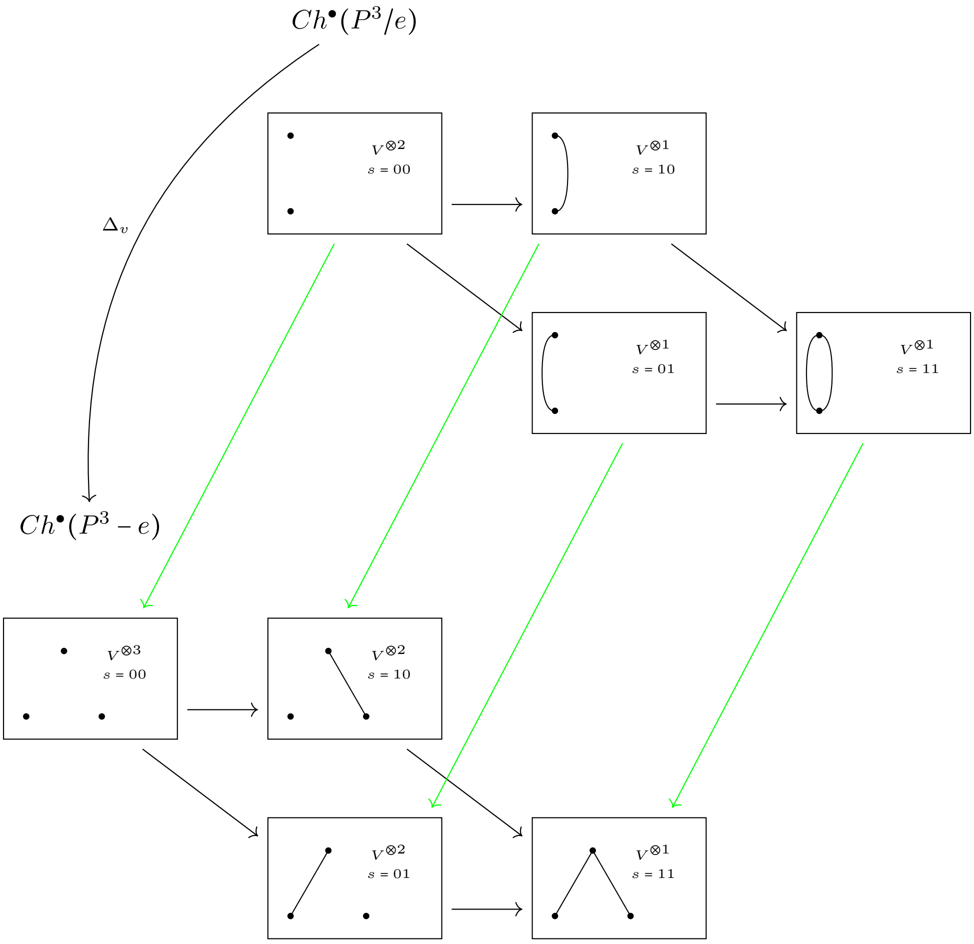
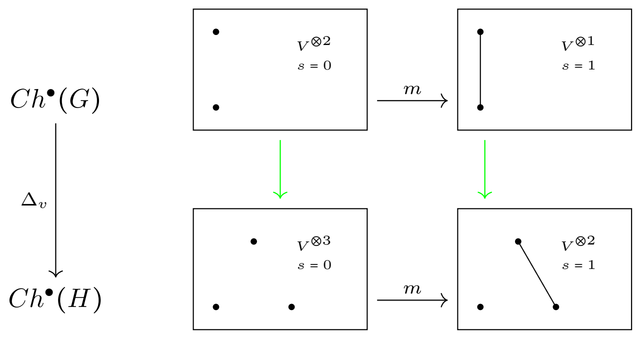


Figure 6: The comultiplication map represents the splitting of a vertex. The highlighted edges represent the comultiplcaiton map on the level of resolutions

because our multiplication and comultiplication structures come from a Frobenius algebra. The geometric intuition follows from using cobordisms instead of these maps described here (and from similar constructions in Khovanov homology– [11] has a good explanation of the functorality of Khovanov homology) but as a topological theory has not been built up for the chromatic complex, we have to do this proof by hand.
It suffice to show that Δ_v commutes with the unsigned differential of the complex– which is just the sum of the edge maps. Therefore, we just need to show that the comultiplication map commutes with the edge maps.

As the commutativity holds true whenever an edge is added to a connected component that does not contain v, we really only need to check for this diagram below.



The commutativity of this diagram comes from a simple computation: we check the most difficult case here. $\Delta m(1 \otimes x) = 1 \otimes x + x \otimes 1$ while $m(\Delta(1 \otimes x)) = (m(1 \otimes 1) \otimes x) + (m(1 \otimes x)) \otimes 1 = 1 \otimes x + x \otimes 1$ ∎

Corollary 2.12 Let G be a graph, and v be a vertex. Then there is an isomorphism $Ch^\bullet(G \cup \{v\}) \to Ch^\bullet(G)$

2.0.7 Unit and Co-unit

This is really a special case of the union map that we had above, but it is so important that it is worth its own treatment. Given a graph G, we define the unit map $i_v : G \to G \cup \{v\}$. This descends to a map on the chain complex defined as follows. Notice that G and $G \cup \{v\}$ have all the same resolutions, except that $G \cup \{v\}$ has an additional connected component. Given an enhanced state (s, c) of G, we define $i_v(S, C)$ to be the enhanced state where the component $\{v\}$ is labelled with a 1.
Likewise, we define the co-unit map $\epsilon_v : G \cup \{v\} \to G$ as follows. Given an enhanced state (s, c) of $G \cup v$, we define $\epsilon_v(s, c)$ to be

- The enhanced state with the same colorings if the labeling of $\{v\}$ is x
- 0, if the enhanced state the labeling of $\{v\}$ is 1.

2.0.8 Short Exact Sequence of Chromatic Homology

When we were talking about the chromatic polynomial, we had a nice inductive definition for the chromatic polynomial which was

$$P_G(\lambda) = P_{G-e}(\lambda) - P_{G/e}$$

We can upgrade this to a relation on the chromatic complex below.

Theorem 2.13 [5] Let G be a graph. Then we have the short exact sequence of chain complex

$$0 \longrightarrow Ch^\bullet(G/e) \xrightarrow{i} Ch^\bullet(G) \xrightarrow{\pi} Ch^\bullet(G - e) \longrightarrow 0$$

Proof. In fact, we have an even stronger result. We can prove that $Ch^\bullet(G)$ is the mapping cone of $m : Ch^\bullet(G/e) \to Ch^\bullet(G - e)$, where m is the multiplication map. Writing this proof out using states takes a while, but is probably best seen as follows. The complex $Ch^\bullet(G) = Ch^\bullet(G/e) \oplus Ch^\bullet(G - e)$, with differential

$$\begin{pmatrix} d_1 & m_{v_1 v_2} \\ 0 & d_2 \end{pmatrix}$$

where d_1 is the differential on $Ch^\bullet(G/e)$ and d_2 is the differential on $Ch^\bullet(G-e)$. The map $m_{v_1 v_2}$ is given by the highlighted arrows in Figure 7 (which should also give intuition for where all the maps are coming from)

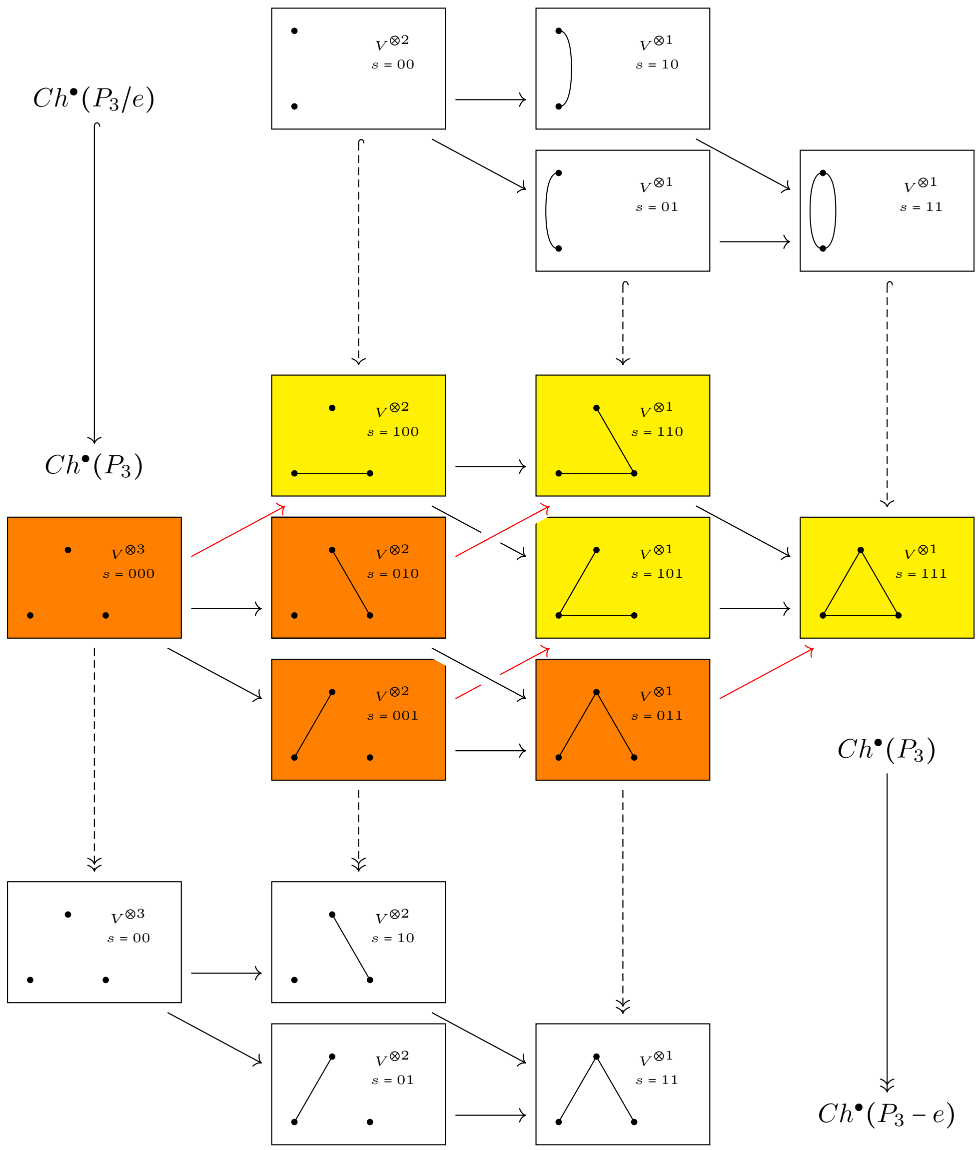


Figure 7: An exact sequence exhibiting that $Ch^\bullet(G)$ is the mapping cone of its contraction and edge removal. The highlighted maps represent the map of complexes giving rise to the mapping cone.

■

Corollary 2.14 There is a long exact sequence on homology

$$\cdots \longrightarrow H^{k+1}(G-e) \longrightarrow H^k(G/e) \longrightarrow H^k(G) \longrightarrow H^k(G-e) \longrightarrow H^{k-1}(G/e) \longrightarrow \cdots$$

3 Topological Interpretation

The goal of this section is to take the above combinatorial definitions of operations between morphism, and reason why topologically these are the logical operations to talk about. The work in this section is largely based on [1]. In Khovanov homology, the complex $Kh^{\bullet}(L)$ is a functor from the category of cobordisms over knots to abelian groups. The data encoded by a cobordism is not enough to specify the data of a graph. I think in general the proper tool to talk about these things are stratified spaces, but since I am working on such a specific case here, I've gone ahead and defined what morphisms between graphs should be without using this machinery, by developing banded cobordisms (See Figure 8 for where these ideas are coming from). A banded cobordism is like a cobordism between graphs, except that it requires that the number of edges in the graph be preserved at all time, and it doesn't use manifolds.

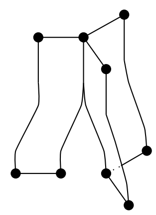


Figure 8: An intuition for how cobordisms of graphs should work

I have several goals for this section:

- Provide a way for composing banded cobodisms
- Define "equivalence of banded cobordisms, " and define generators for these equivalence classes
- Show that banded cobordisms under the above equivalence define a category.
- Define a TQFT-like functor from the category of banded cobordisms to graphs.
- Show that there is a natural interpretation of the Chromatic complex in the abelianization of the category of banded cobordisms.

I'm not sure if I will make it through all of these items before the due date of this paper, but I'll try my best. There may be some unproven claims at the end.

3.1 Banded Cobordisms

Let G be a graph and H be two different graphs. Then a **banded cobordism** (or b-cobordism for short) between G and H is a pair (B, W) where

- B is a graph. Then these is a inclusions of graphs $\psi_G : G \to B$ and $\psi_H : H \to B$.
- A **face set** W is a collection of circuits (can repeat vertices, not edges) in B which satisfies the following properties
  - Every circuit $C_i \in W$, contains one edge of G and one edge H.
  - Every edge of G and every edge of H is contained in exactly one C_i.

This definition looks rather strange, but it's designed to capture the combinatorial properties of a cobordism. We could create a CW complex out of the data from (W, B) by attaching disks to the specified circuite C_i. As a result, We will will call the subgraph $\underline{\mathrm{B}} := B - (E(G) \cup E(H))$ the set of **bands** on the cobordism, and this provides us data on how the vertices of a graph are effected by a banded cobordism. It's probably best to see what qualifies as a banded cobordism by looking at a few examples of things that are (Figure 9) and are not (Figure 10) banded cobordisms.

Banded cobordism carry "Feynman Diagram" type data for vertices, and "Cobordism" type data for the edges of a graph. We will denote a banded cobordism $(W, B) : G \Rightarrow H$, in the style of [8]. The reason that we are going to use b-cobordisms as morphisms between graphs is becaues they capture the comultiplication and multiplication structure that we developed earlier.

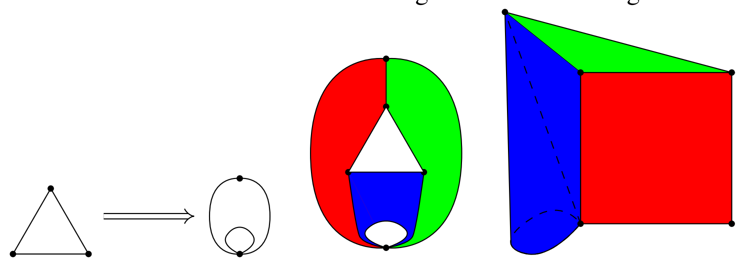


Figure 9: An example of a b-cobordism. Here I have colored the face to represent the sets C_i in W

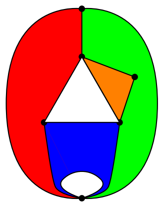


Figure 10: A non-example of a b-cobordism

Claim 3.1 Let G and H be graphs.

- Suppose that there is a graph isomorphism $f : G \to H$. Then there is a banded cobordism $\bar{f} : G \to H$.
- Suppose that there is a splitting $\Delta_{E_1 E_2} : G \to H$. Then there is a banded cobordism $\bar{\Delta}_{E_1 E_2} : G \Rightarrow H$
- Suppose that there is a merging $m_{v_1 v_2} : G \to H$. Then there is a banded cobordism $\bar{m}_{v_1 v_2} : G \Rightarrow H$.
- Suppose that there is a unit mapping $i : G \to H$. Then there is a banded cobordism $\bar{i} : G \Rightarrow H$.
- Suppose that there is a counit mapping $\epsilon : G \to H$. Then there is a banded cobordism $\bar{\epsilon} : G \Rightarrow H$.

An exact exhibition of these constructions will be left off until Section 3.2. Like cobordisms, we have a way to compose banded cobordisms.
Let $(B, W) : G_1 \Rightarrow G_2$, and $(B', W') : G_2 \Rightarrow G_3$. Then we get a b-cobordism $(C, V) = (B, W) \circ (B', W') : G_1 \Rightarrow G_3$.

- For the graph C, glue B and B' together by the inclusions $\psi_{G_2} : G_2 \to B$ and $\psi'_{G_2} : G_2 \to B'$. Then remove all the edges of G_2 from $B \cup_{G_2} B'$. So $C = (B \cup_{G_2} B') - E(G_2)$.
- For every edge $e \in E(G_2)$, there are cycles $C_e \in W$ and $C'_e \in W'$ which contain e. Define $D_e = C_e \cup C'_e - \{e\}$. This is circuit in C. Define $V = \{D_e\}_{e \in G_2}$

One can check that this gives (C, V) the structure of a banded cobordism. Figure 11 shows how one can put together two cobordisms pictorially.

We can see where this should be going— we should be able to compose b-cobordisms and make a category out of these things. Our goal should be to develop a category with

- Objects are given by graphs
- Morphisms are b-cobordism $(C, V) : G \Rightarrow H$
- Composition is b-cobordism composition.

As defined right now, banded cobordisms give us a semicategory. To upgrade this structure into a cateogry, we are going to add in some equivalence to make the notion of identity b-cobordism make sense. In traditional cobordisms, the natural equivalence relation to put on cobordisms is simply diffeomorphisms.

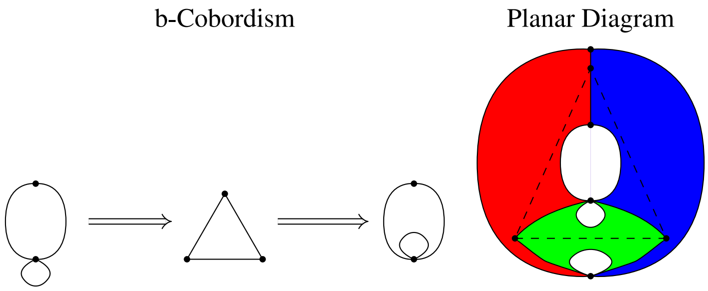


Figure 11: The b-cobordism composition on the left is represented by the diagram on the right. The dotted line represent the triangle that is removed during the composition. The shaded regions represent the faces of the cobordim.

We would hope that the CW-structure that we could put on b-cobordisms could give us insight on how to define the equivalence relation between b-cobordisms– something like " two b-cobordisms are equivalent if their associated CW-structures are homeomorphic". However, there are b-cobordisms that we will want to be equivalent, but are not homeomorphic 12. Our definition of equivalence is going to be based on the

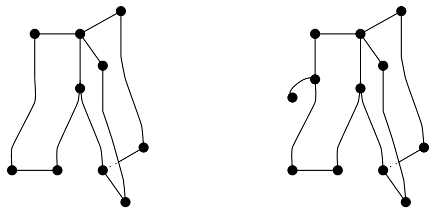


Figure 12: Two b-cobordisms that we would like to be equivalent, but are not homeomorphic

combinatorial structure of the b-cobordism. An equivalence of b-cobordisms should preserve the face set structure of the b-cobordism, but changing the bands on the cobordism shouldn't be considered a topological property of the b-cobordism. The type of changes to the bands of a cobordism that we'll consider are expansions and contractions.

Definition 3.2 Let $(B, W) : G \Rightarrow H$ be a b-cobordism. For any tree of $T \subset B - E(G) - E(H)$ (that, for technical reasons, cannot contain both vertices from G or H) define the set $W/T := \{C_i/T \subset B/T | C_i \in W\}$. Then define the **contraction of** (B, W) **along** T to be the b-cobordism $(B/T, W/T) : G \Rightarrow H$. The reverse operation is called an **expansion**.

Our equivalence relation between cobordisms is that two b-cobordisms are the same if they can be connected by a string of expansions and contractions. We can make this definition a little more compact if we use the following definition instead.

Definition 3.3 Let $(B, W), (B', W') : G \Rightarrow H$ be two b-cobordisms. Then we say that (B, W) and (B', W') are **equivalent** (and write $(B, W) \simeq (B', W')$) if there is a series of expansions and contractions taking (B, W) to (B', W').

From the definition it is immediate that b-cobordim equivalence is an equivalence relation.

Lemma 3.4 b-cobordism equivalence is respect cobordism composition. That is, if $(B, W) \simeq (B', W') : G \Rightarrow H$ and $(C, V) \simeq (C', V') : H \Rightarrow J$, then $(B, W) \circ (C, V) \simeq (B', W') \circ (C', V') : G \Rightarrow J$.

Theorem 3.5 $b\mathcal{C}ob$, the which has graphs as objects and equivalence classes of b-cobordisms as morphisms, is a category.

3.2 Generators of $b\mathcal{C}ob$

The goal of this section is to show that the category of banded cobordisms enjoys many of the same structural properties as the category of 2d-cobordism. In [8], 2d-cobordisms, Feynmann diagrams and Frobenius algebras are all categorized as structures that have the operations of merging, splitting, creation and annhilation (See Table 1)
b-cobordisms do not share a certain nice property of 2d-cobordisms, Feynmann diagrams and Frobenius

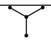
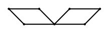
Principle	Feynmann Diagram	2D Cobordism	Algebraic Operation	b-Cobordism
Merging		□	$m : V \otimes V \to V$	
Creation		□	$i : R \to V$	
Splitting		□	$\Delta : V \to V \otimes V$	
Annhilation		□	$\epsilon : V \to R$	

Table 1: A common theme in cobordism-like structures. Based on [8]

algebras. In all of those categories, the objects being annhihlated, merged, etc. are all the same. A Feynmann diagram is a method of keeping track of operations of vertices. A Cobordism is a way to keeping track of operations on circles. In our case, b-cobordisms keep track of these types of operations of graphs (of which there are many kinds, as opposed to circles or vertices.)
In 2d-cobordisms, there is a theorem that says that every cobordism is a composition of these 4 types that we have listed above. It will be out goal to show that every b-cobordism of graphs is a composition of multiplication and comultiplication maps. To get started, we will prove Lemma 3.1 by explicitly exhibiting some cobordisms.

Definition 3.6 An **elementary banded cobordism** is any of the following 5 types of b-cobordisms.

- Given two graphs G and H, and a graph isomorphism $G \to H$, we can construct cobordism $\bar{f} := (B, W) : G \Rightarrow H$ as follows. The graph B is $G \sqcup H$ with additional edges drawn between v and $f(v)$ for every $v \in V(G)$. For every $e \in E(G)$ with endpoints v, w, the corresponding face C_e in W is given by the cycle $v - -w - -f(w) - -f(v) - -v$ in B. Note the special case of the identity cobordism $\bar{\text{id}} : G \to G$
- Given a graph G, and a merging $m_{v_1 v_2} : G \to G \cdot (v_1 v_2)$, we can construct the merging banded cobordism $\bar{m}_{v_1 v_2} := (B, W) : G \Rightarrow G \cdot (v_1 v_2)$ as follows. Let $(B', W') = \bar{\text{id}} : G \to G$ be the identity cobordism. Let v_1', v_2' be the vertices of v_1, v_2 identified the image boundary of $\bar{\text{id}}$. Define a new cobordism $\bar{m}_{v_1 v_2} := (B, W)$ by taking $B' = B \cdot v_1 v_2)$ and $W = \{C_i \cdot (v_1 v_2)\}$.
- Given a graph H and a splitting $\Delta_{E_1 E_2} : G \to G \div (E_1 E_2)$, we can construct a splitting b-cobordism $\bar{\Delta}_{E_1 E_2} := (B, W) : G \Rightarrow G \div (E_1 E_2)$ by taking the merginging cobordism and "running it backwards."
- Given a graph G, and a unit map $i : G \to G \cup v$, we have a b-cobordism $\bar{i}G \Rightarrow G \cup v$. This is the identity cobordism with an additional vertex added.
- Given a graph G and a counit map $\epsilon : G \to G \backslash v$, we have a b-cobordism $\bar{\epsilon} : G \Rightarrow G \backslash v$. This is the unit cobordism in reverse.

We want to show that these are the building blocks of all graph homomorphism. First, we prove a quick criteria that we will use to decompose b-cobordism.

Lemma 3.7 Let $(B, W) : G \Rightarrow H$ be a b-cobordism. Suppose there exists a map $\phi : V(G') \to V(\bar{B})$ such that

- The image of G' is disjoint from both G and H in $\bar{B}$
- $|E(G')| = |E(G)| = |E(H)|$

- If $e \in G'$ has endpoints v, w, there exists a band $C_i \in W$ sucht that $\phi(v)$ and $\phi(w)$ belong to C_i.

Then the b-cobordism (B, W) factors through G'.

Now, the main theorem that shows that all we really need to understand are the elementary b-cobordisms.

Theorem 3.8 Let $(B, W) : G \to H$ be a banded cobordism. The (B, W) is equivalent to the composition of elementary banded cobordisms.

Proof. Let (B, W) be a cobordism, and let

- $\alpha(B, W) = |E(\bar{B})| - |V(H)|$ be the number of edges in the bands of the cobordism less the number of vertices in H
- $\beta(B, W) = \left(\sum_{v \in B \cap \phi(G)} \deg_B(v)\right) - 2|E(G)| - |V(H)|$. This is like the number of edges in the bands of the cobordism that contain a vertex of G, but are not contained in the inclusion of G into B, less the number of vertices in H.

If $\alpha(B, W) = \beta(B, W) = 0$, then we must have the identity cobordism between G and H. We will prove that given a non-identity b-cobordism $(B, W) : G \Rightarrow H$ that we can produce an elementary cobordism $(E, V) : G \to G'$ and a new cobordism $(B', W') : G' \to H$ such that either

- $\alpha(B, W) > \alpha(B', W')$ and $\beta(B, W) \geq \beta(B', W')$.
- $\beta(B, W) > \beta(B', W')$

From here, the proof involves a lot of keeping track of details on how to exactly lower α and β, but the general idea is that whenever we have $\beta > 0$, we can simplify the b-cobordism by factoring it through a comultiplication map, and whenever $\beta = 0$ and $\alpha > 0$, then we can find a multiplication map somewhere in the cobordism. When both α and β are zero, a cobordism is the identity with some additional units and counits attached. So, let us break into those cases.

- Suppose that $\beta > 0$. Then there is a vertex of w in $G \in B$ that has two edges in $\bar{B}$. Select one of those edges e_x. Instead of looking at (B, W), look at the expansion $(B, W) \circ I$, where $I : G \to G$ is the identity cobordism. For every vertex $v \in G$, there exists a unique edge f_v in the bands of $(B, W) \circ I$ that contains v. Define v' to be the vertex of $(B, W) \circ I$ that is the endpoint of f_v not contained in G. We will now perform an expansion on w'. Attached to w' two edges e'_x and e'_y . Expand w' to an edge between u, u' so that e'_x and f_w are joined to u, and every other edge of $(B, W) \circ I$ that contains w is attached to u'. Subdivide the edge e'_x in two. Now contract the edge f_w. Call this new b-cobordism (B'', W'') We have a map $\phi(V(\Delta_w(G)))$ into (B'', W'') that satifies the conditions of the above lemma. So there is a factorization through $\Delta_w(G)$, that is $(B, W) = (B', W') \circ \Delta_w$. This process does not rais the number of edges between (B, W) and (B', W') and lowers the $\beta(B, W)$.
  While this type of reduction is a bit hard to follow symbolically, Figure 13 provides a diagram of how it works.
- Suppose that $\beta = 0$. Then for every vertex $v \in V(G)$, we have that there is unique $e_v \in \bar{B}$ that contains v. We do the easy cases now:
  - Suppose for all $v \in V(G)$, the other endpoint of e_v lies in H. Then $\alpha = 0$
  - Suppose there is a vertex v such that e_v shares no endpoints with e_w for all $w \neq v$ and e_v's other endpoint is not in W. Then we can contract e_v to get an equivalent b-cobordism with lower α value

  In all other cases, we can find vertices $x, y \in V(G)$ such that e_x and e_y share a common endpoint z. Again, look at $(B, W) \circ I$, where I is the identity. To every vertex $v \in V(G)$, we have an edge f_v in the bands of $(B, W) \circ I$ that contains v. Let v' the endpoint of f_v not contained in G. Contract the edges f_x and f_y. Then on this cobordism, we have a map from $\phi(V(m_{xy}(G))$ to this cobordism that satisfies the condition of the lemma above. So (B, W) factors as $(B', W') \circ m_{xy}$ which lowers $\alpha(B, W)$.

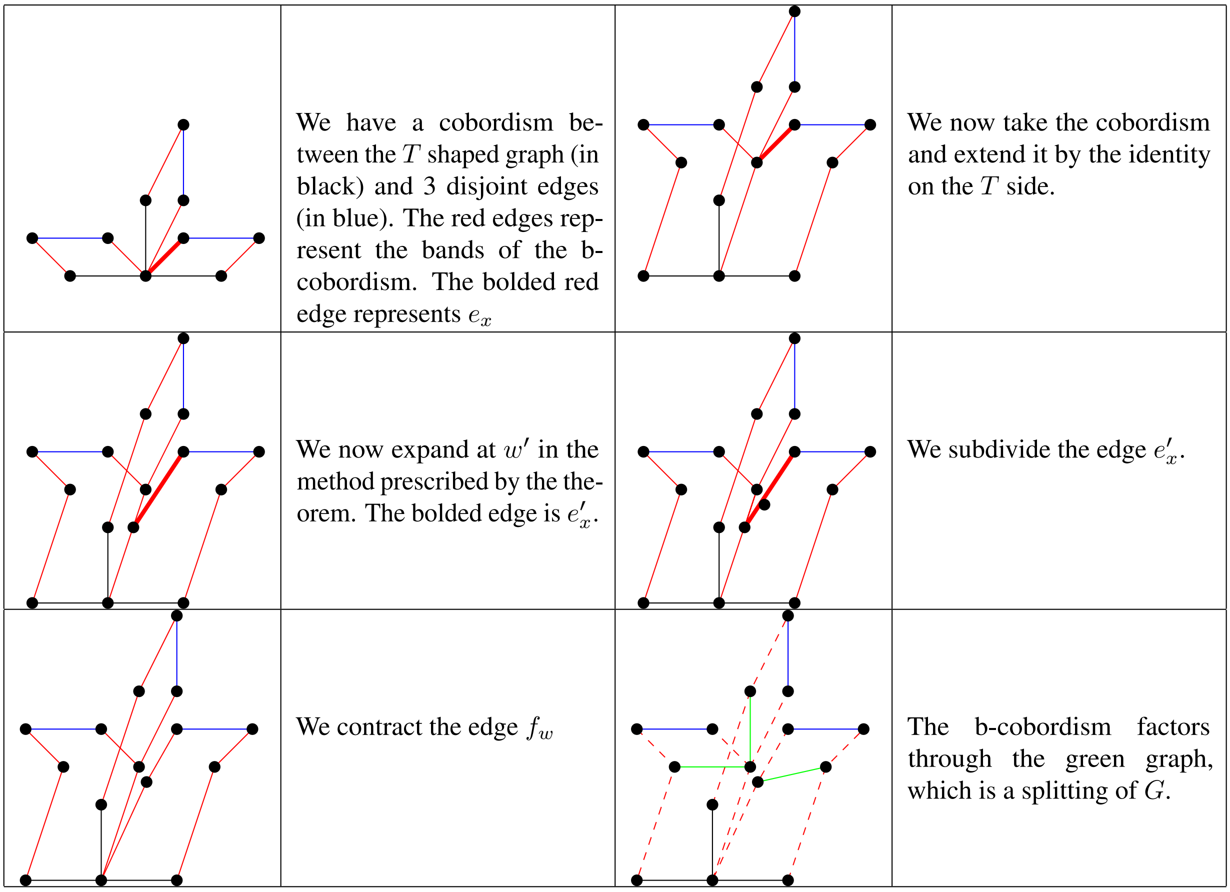


Figure 13: An example of the reduction

■

This shows that every single b-cobordism can be written as the composition of many elementary cobordisms.

3.3 Relations in $b\mathcal{C}ob$ and cannonical form

To give a complete characterization of the category $b\mathcal{C}ob$, we need to not just know generators for morphisms in the category, but also give explicit relations between those generators. In other words, we want to find expressions for the relations given by b-cobordism equivalence in terms of the elementary generators.

3.3.1 Removing Unit and Counit

The first relation that in a connected b-cobordism we can remove units and counits. We have the following relations.

- (Counit removal) Let E be the edges of a vertex v, and let $\Delta_{E\varnothing}$ be the b-cobordism that splits v into two new vertices, v_1 and v_2. Then $\pi_{v_1}\Delta_{E\varnothing} = \text{id}$.
- (Unit Removal) Let i_v be the b-cobordism inclusion of a new vertex. Then let w be any vertex. We have $m_{vw}i_v = \text{id}$.

We skip a proof of these two relations, and refer to Figure 14 as justification

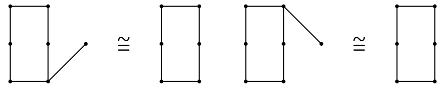


Figure 14: Removing the Unit or Counit

3.3.2 Associativity of Multiplication and Comultiplication

Multiplication and comultiplication are “ associative” operations, meaning that the order by which you join things together doesn’t matter, nor does the order that you split them apart. More precisely

- Let v_1, v_2, v_3 be three different vertices of a graph G. Suppose that $m_{v_1 v_2}$ joins v_1 and v_2 to a new vertex w_{12}, and $m_{v_2 v_3}$ joins v_2 and v_3 to a new vertex w_{23}. Then we have the identity

$$m_{v_3 w_{12}} m_{v_1 v_2} = m_{v_1 w_{23}} m_{v_2 v_3}$$

- Let $E_1 \sqcup E_2 \sqcup E_3$ be a partition of the E containing v as an endpoint. Then

$$\Delta_{E_1 E_2} \Delta_{(E_1 \cup E_2) E_3} = \Delta_{E_2 E_2} \Delta_{E_1 (E_2 \cup E_3)}.$$

Again, we skip the proof of these relations and refer to Figure 15 as justification

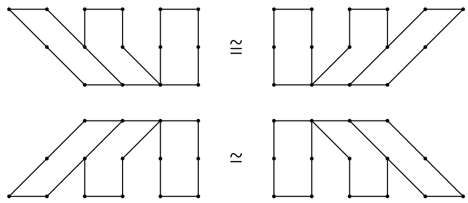


Figure 15: Associativity of Multiplication and Comultiplication

3.3.3 The Frobenius Relation

The Frobenius relation states that multiplication and comultiplication “commute” with eachother provided that they don’t share the same set of vertices.

Lemma 3.9 **Frobenius Relation**

Let v be a vertex of G. Suppose that $E_1 \sqcup E_2$ as a partition of E, the set of edges that contain v. Let $\Delta_{E_1 E_2} : G \Rightarrow G \div E_1 E_2$ be the splitting at v that takes v to v_1 and v_2. Let w be any vertex not equal to v_1 or v_2 two vertices that are not both v_1 and v_2. Let E' be the edges containing v as a vertex. Then in the graph $G \cdot (vw)$ let v' correspond to where v and w were joined together. Let $F_1 = E_1$ and $F_2 = E_1 \cup E'$ be a partion of $F = F_1 \sqcup F_2$, where F are the edges that contain v'. Then

$$m_{w_2 v} \circ \Delta_{E_1 E_2} = \Delta_{F_1 F_2} \circ m_{vw}$$

Proof. This identity is a little tricky to visualize, so we should draw a picture first (Figure 16). Take

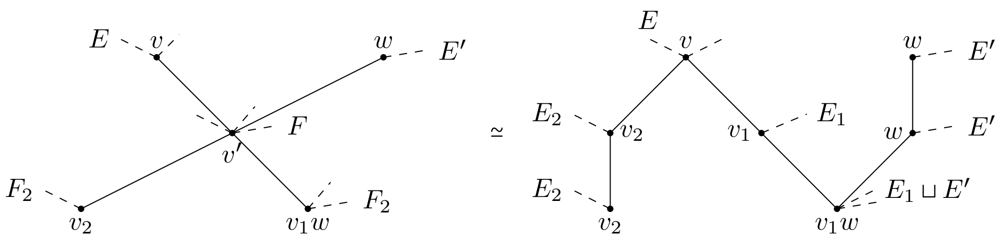


Figure 16: The Frobenius Relation

$\Delta_{F_1F_2} \circ m_{vw}$, and expand it on both sides by the identity. Then take the tree traced out by $v_2 - v - v_1 - w$ and contract it. I've marked these edges in bolded red in Figure 17. ■

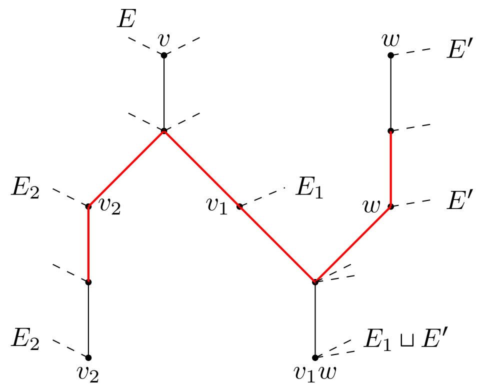


Figure 17: Contract these bolded red edges to prove the Frobenius relation.

3.3.4 Sufficency of relations

We now use the above relationships to give a cancellation way of writing any b-cobordism as a composition of elementary b-cobordisms. Before we present the cannonical form, it will be convenient to have the following notation:

Definition 3.10 Let v be a vertex of G, $E' \sqcup E''$ be a partition of E, the edges containing v, and $\Delta_{E'E''}$ split v into v' and v''. Then define $T_{E'_v} := m_{v'v''}\Delta_{E'E''}$. (call this a hole)

Lemma 3.11 Let $E' \neq F'$ be two subset of E, the edges containing v. Then $T_{E'_v}T_{F'_v} \nsim T_{F'_v}T_{E'_v}$.

Theorem 3.12 Suppose that $(B, W) : G \to H$ be given two different decompositions as elementary cobordisms,

$$\begin{aligned}(B, W) =& x_1 \circ x_2 \circ \cdots x_n \\ (B', W') =& y_1 \circ y_2 \circ \cdots y_m\end{aligned}$$

So that as b-cobordism, (B, W) and (B', W') are equivalent. Then only using the relations of elementary cobordisms described earlier in this section, we can rearrange the elementary b-cobordisms of the second decomposition to be the first one.

Proof. Start by looking at the first elementary cobordism in the decomposition of (B, W). We will use only the relations of elementary cobordisms to produce a (B'', W'') from (B', W') with the property that it's first elementary cobordism is the same as the first elementary cobordism of (B, W). We break into cases based on what type of elementary cobordism x_1 is.

- Suppose that the first elementary b-cobordism in (B, W) is a multiplication b-cobordism, m_{vw} where v, w are vertices of G. Let e_v, e_w be edges connected to v and w in G. If C'_e and C'_{e_w} are the two cycles in W associated to e_v and e_w, let P'_{e_v} and P'_{e_w} be the be the paths that start at v and w in G and travel to the graph H along the cycle C'_{e_v} and C'_{e_w}. Notice that these paths have length m, so each edge of the path corresponds to one elementary cobordism in the decomposition of (B', W').

Claim 3.13 The paths P'_{e_v} and P'_{e_w} intersect.

Proof. Let P_{e_v}, P_{e_w} be paths associated to v and w in the cobordism (B, W). Then P_{e_v} and P_{e_w} intersect after their very first edge. Suppose we take (B, W) and expand or contract it by a single edge to get (B'', W''). Then P''_{e_v} and P''_{e_w} also intersect. So B cobordism equivalence preserves the intersection of these paths. ∎

Suppose that P'_{e_v} and P'_{e_w} first intersect at the y_k elementary cobordism. Then we can "push" that elementary cobordism to the left via the relation of associativity of multiplication and the Frobenius relation. The only case where we cannot apply the frobenius relation is if the the cobordism preceding y_k is a comultiplication that splits a vertex into v and w. However, this would mean that P'_{e_v} and P'_{e_w} intersect at a place before k, contradicting the minimality of k. Therefore, we can rearrange the elementary decomposition to give us $(B'', W'') = x_1 \circ y'_1 \circ \cdots y'_{m-1}$.

- Suppose that the first elementary cobordism in the decomposition is a comultiplication b-cobordism, $\Delta_{E_1 E_2}$. Then let $e_1 \in E_1$, $e_2 \in E_2$ be two edges. Look at the corresponding paths P'_{e_1} and P'_{e_2} in (B', W'). Then a proof similar to the above claim proves that P'_{e_1} and P'_{e_2} are not the same path. Let k be the first place where these two paths differ. Then the elementary cobordism y_k can be moved to the left via the relation of associaticitivity of comultiplication and the Frobenius relation. Then only time that we wouldn't be able to do this is if the elementary cobordism that proceeded y_k was a multiplication. However, in this case, it means that P'_{e_1} and P'_{e_2} began differing at some point before k, contradicting the minimalility of k. Therefore we can rearrange the cobordism to be of the form $(B'', W'') = x_1 \circ y'_1 \circ \cdots y'_{m-1}$.
- Similar arguements show that units and counits can be moved to the top.

Having produced such a (B'', W'') from (B', W') via the elementary relations, we can proceed using induction to rearrange all of the elementaray cobordisms of (B', W') to match those of (B, W). ∎

Corollary 3.14 The relations of associativity, cancellation of unit and counit, and the Frobenius relations generate b-cobordism equivalence.

Remark. The proof above is a little bit messy in notation. The elementary decomposition that we used to break down the b-cobordism is very similar to a handlebody decomposition that we could use to break down a regular cobordism using morse theory. Is there a slicker proof of the above statement using discrete Morse theory? Here are some correspondances between the two theories that make me think so

b-cobordisms	cobordism
b-cobordism equivalence	smooth homotopy
producing an elementry decomposition	producing a morse function
elementary b-cobordisms	critical points
paths P_{e_v}	flow lines with respect to some morse function

3.4 Functors

We now give a functor from the category of b-cobordisms to the category of R algebras. Recall we had defined earlier

Definition 3.15 A **Frobenius Algebra** V is a $\mathbb{R}$ algebra equipped with the following maps

$$m : V \otimes V \to V \qquad i : R \to V$$
$$\Delta : V \to V \otimes V \qquad \epsilon : V \to R$$

such that the following diagrams commute

$$\begin{tikzcd} A \otimes A \arrow[r, "\Delta\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m"] & A\otimes A\otimes A \arrow[d, "\mathrm{id}\otimes m"] \\ A \arrow[r, "\Delta"] & A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "\mathrm{id}\otimes\Delta"] \arrow[d, "m\otimes \mathrm{id}"] & A\otimes A\otimes A \arrow[d, "\mathrm{id}\otimes m"] \\ A \arrow[r, "\Delta"] & A \otimes A \end{tikzcd}$$

Definition 3.16 A **TQFT** is a functor from the category of b-cobordisms to the category of R-algebras.

Let's go and actually construct one of these functors. Let V be a Frobenius algebra over R. Define the functor $\mathcal{F} : b\mathcal{C}ob \to R\mathrm{alg}$ as follows. To every graph G, $\mathcal{F}(G) = \bigotimes_{x\in K(G)} V_x$, where $|K(G)|$ is the number of connected components of G. In order to define the values of $\mathcal{F}$ on cobordism, we need only know where $\mathcal{F}$ takes elementary cobordisms, and check that it respect the relationship of cobordism equivalence.
Define the values of $\mathcal{F}$ on elementary cobordisms as follows

- If $m_{vv} : G \to H$ is an elementary cobordism which merges two connected components together, then let $\mathcal{F}(m) : \mathcal{F}(G) \to \mathcal{F}(H)$ be that takes $V_v \otimes V_{v'} \to V_w$ by the multiplication map m of Frobenius algebras. Otherwise, let $\mathcal{F}(m) : \mathcal{F}(G) \to \mathcal{F}(H)$ act by the identity.
- If $\Delta_{EE'} : G \to H$ is an elementary cobordism which splits one connected component into two, then let $\mathcal{F}(\Delta) : \mathcal{F}(G) \to \mathcal{F}(H)$ splitting w into v and v' be that which takes $V_w \to V_v \otimes V_{v'}$.
- If $i : G \to H$ is the inclusion of an additional point into H, then define $\mathcal{F}(i) : \mathcal{F}(G) \to \mathcal{F}(H)$ to be the unit map of Frobenius algebras.
- If $\epsilon : G \to H$ is the removal of a connected component, let $\mathcal{F}(\epsilon) : \mathcal{F}(G) \to \mathcal{F}(H)$

Since every b-cobordism can be decomposed into a a series of elementary cobordism, we can define the value of $\mathcal{F}$ on $(B, W) = x_1 \circ \cdots \circ x_k$ to be $\mathcal{F}(x_1) \circ \mathcal{F}(x_2) \circ \cdots \circ \mathcal{F}(x_k)$. To show that this map is well defined, we need it to be independent of elementary decomposition.

Theorem 3.17 $\mathcal{F}(B, W)$ is independent of elementary decomposition chosen for (B, W).

Proof. It suffices to check that $\mathcal{F}$ respects the elementary cobordism relationships defined in the earlier section, as we showed that these relationships generated all of the b-cobordism equivalences. The suggestive names show that required relationships hold— i.e. associativity of b-cobordism multiplication corresponds to associativity of multiplication in Frobenius algebras. ■

Corollary 3.18 $\mathcal{F} : b\mathcal{C}ob \to R\mathrm{alg}$ is a functor.

Remark. The b-cobordism relations are strictly stronger than the Frobenius algebra relations, making $\mathcal{F}$ a forgetful functor. We can still include the monoidal category of frobenius algebras into the category of b-cobordisms by sending each forbenius algebra $V^{\otimes n}$ to the graph with n vertices. Let's call this functor $\mathcal{G} : \mathrm{Frob} \to b\mathcal{C}ob$. Then $\mathcal{F}\mathcal{G}$ is the identity, while $\mathcal{G}\mathcal{F}$ is very much not the identity. This shows that the category of $b\mathcal{C}ob$ contains a lot more data than frobenius algebras, (and equivalently the category of Feynman diagrams or 2-cobordisms.)

3.5 The Chromatic Complex, Revisited

The ideas in this section are almost completly lifted from Bar-Natan's paper on cobordisms. [1].

3.5.1 Abelianization of the Category of Banded Cobordism

A representation of Khovanov homology due to Bar-Natan in [1] treats the Khovanov complex as a complex of cobordisms. This representation of the Khovanov complex is universal in the sense that other representations

ofthe Khovanov complex can be realized by applying TQFTs to this complex. In this section, we will outline the methods that he used in order to construct the chromatic complex in the category of $b\mathcal{C}ob$. Our plan will consist of constantly "upgrading" our category to have the required structures to do homology.
Our first upgrade will be taking the category $b\mathcal{C}ob$ and making it a pre-abelian category. Define the category $b\mathcal{C}ob^+$ to have the same objects as $b\mathcal{C}ob$, and whose morphisms are formal $\mathbb{Z}$ linear combinations of the morphisms of $b\mathcal{C}ob$, and whose composition laws are defined via bilinearity. This automatically makes $b\mathcal{C}ob^+$ an pre-additive category. While $b\mathcal{C}ob^+$ is a pre-additive category, we want to upgrade it to an additive category. We do this by formally extending the objects of $b\mathcal{C}ob$ to formal direct sums of the objects that we already have.

Definition 3.19 [1] If $\mathcal{A}$ is a preabelian category, we can define its **additive closure** Mat$(\mathcal{A})$ as follows.

- The objects in Mat$(\mathcal{A})$ are formal direct sums of objects in $\mathcal{A}$. So an object is of the form $\bigoplus_i M_i$ for some objects M_i of $\mathcal{A}$.
- The morphisms of Mat$(\mathcal{A})$ are matrices of morphisms from $\mathcal{A}$. Formally, $(A_{ij} : \bigoplus_i M_i \to bigoplus_j N_j$ is the sum $\sum_{ij} A_{ij}$, where each A_{ij} is a morphism $A_{ij} : M_i \to N_j$.
- Morphisms in Mat$(\mathcal{A})$ can be added and multiplied using the standard rules for matrix multiplication.

One can check that Mat$(\mathcal{A})$ is an additive category. Given an additive category Mat$(\mathcal{A})$, we can look at the cateogry of chain complexes over it, Kom$(\text{Mat}(\mathcal{A}))$, which is the where we will want to eventually end up. Right now, we have the following "upgradings" of categories:

$$b\mathcal{C}ob \Rightarrow b\mathcal{C}ob^+ \Rightarrow \text{Mat}(b\mathcal{C}ob^+) \Rightarrow \text{Kom}(\text{Mat}b\mathcal{C}ob^+)$$

As this last category is a bit too long to write down, we will call it from now on Kb$\mathcal{C}$ob. A more detailed explantion of this construction can be found in [1].
Our goal is to give a functor from the category $b\mathcal{C}ob \to$ Kb$\mathcal{C}$ob which resembles the chromatic complex that we gave in Chapter one. Our second goal after that will be to recover the chromatic complex from this purely topological construct.

3.5.2 The Formal Chromatic Bracket

Given a graph G, we would like to associate to it a chain complex $[[G]]$ in Kb$\mathcal{C}$ob so that the following diagram of categories commutes up to isomorphism

$$\begin{tikzcd} b\mathcal{C}ob \arrow[r, "{[[\cdot]]}"] \arrow[dr, "Ch^\bullet(\cdot)"'] & \text{Kb}\mathcal{C}\text{ob} \arrow[d, "\mathcal{F}_\mathcal{R}"] \\ & \text{Kom}(R) \end{tikzcd}$$

We construct as follows. For every graph G fix an orientation of edges in the graph. For each subset $s \subset E(G)$, define the s resolution of G, $[G : s]$ to be the graph G with a splitting applied at the head of each edge $e \notin s$ into two vertices, v_e and v'_e.Define

$$[[G]]^i = \bigoplus_{|s|=i} G_i.$$

See Figure 18.
If $t = s \cup \{e\}$, define the edge map $d_{st} : [G : s] \to [G : t] = m_{v_e v'_e}$ to be the mergeing that takes the two vertices associated to the head of e. Define the chain map on $[[G]]^i$ to be the signed sum of all of the d_{st}. Then as the multiplication map is associative, we have that $d^2 = 0$ and $[[G]]^i$ is a chain complex.

Claim 3.20 $[[G]]$ is a functor from the category $b\mathcal{C}ob \to$ Kb$\mathcal{C}$ob

Proof. One needs to check that the elementaray morphisms become chain maps in this category. ∎

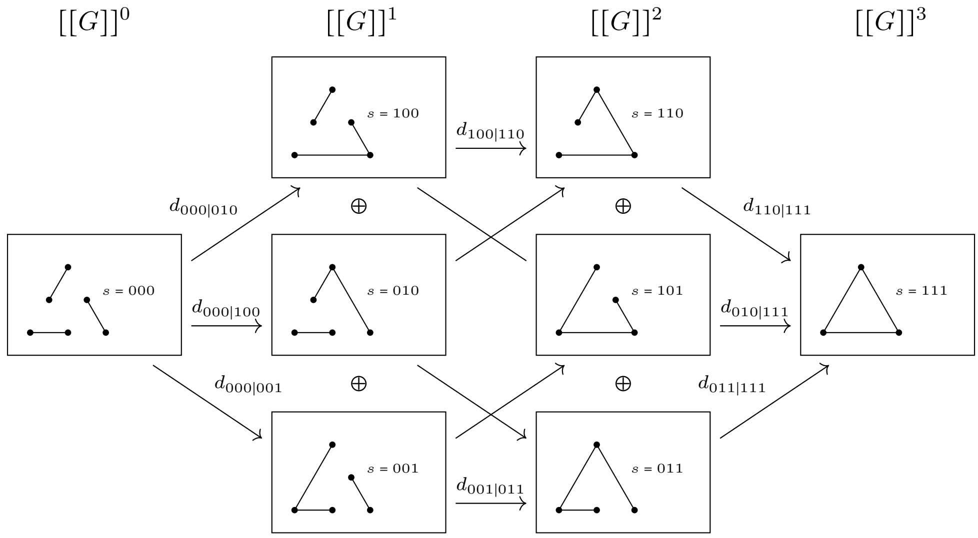


Figure 18: The Chromatic Bracket

Claim 3.21 Let E be the edges attached to v, e be an edge between v and w, $E_1 = \{e\}$, and $E_1 \sqcup E_2 = E$. In Kb$\mathcal{C}$ob, we have the exact triangle $[[G \div E_1 E_2]]^\bullet \to [[G]]^\bullet \to [[(G \div E_1 E_2) \cdot (vw)]]$

Proof. This is exactly the same structure that we saw earlier when we worked with the chromatic complex. ■

Claim 3.22 For the TQFT $\mathcal{F}$ described earlier, we have that $\mathcal{F}([[G]]^\bullet) \cong Ch^\bullet(G)$

Proof. This is clear from the construction. Given a choice of orientation of edges in the graph, there is a natural correspondence between the connected components in th resolutions for $[[G]]$ and the resolutions in the chromatic complex. ■

The only thing that is slightly disappointing about this definition is that it relies on the orientation of the edges of the graph to make the functor work. Since we know that $\mathbb{F}_2[[G]]$ is independent of edge orientation, the homology groups of $\mathbb{F}_2[[G]]$ are independent of edge orientation. However, the choice that we make is a bit disappointing. We might want to instead define a chain theory as follows:

Definition 3.23 Let G be a graph. Define a chain complex $\langle\langle G\rangle\rangle^\bullet$ as follows. For each $s \subset E(G)$, define the $\langle G : s\rangle$ resolution to be the graph G where every edge not in s has been split at both vertices from the complex.

Claim 3.24 $\mathcal{F}(\langle\langle G\rangle\rangle^\bullet)$ produces a chain complex whose homology gives the evaluation of the Tutte Polynomial at $T(1-q, q)$

This has the advantage of being an invariant defined independently of edge oreintation.

4 Things to Look at next

Here is a list of topics that I think could be interesting to look at sometime.

4.1 Reduced Chromatic Homology

This construction is based on Khovanov's construction of reduced Khovanov homology in [7]. Instead of working with a graph G, we will work with a pointed graph (G, p). We first look at the subcomplex of $Ch^{\bullet}(G)$ of elements where the connected component containing p is marked with an x. This can be written as

$$W'_s = \{x\} \otimes V^{k(s)-1}$$

And defining

$$C'^{n}(G) = \sum_{|s|=n} \widetilde{W}_s$$

Then we define the **reduced chromatic complex** to be the quotient

$$0 \longrightarrow C'^{\bullet}(G) \longrightarrow Ch^{\bullet}(G) \longrightarrow \widetilde{Ch}^{\bullet}(G) \longrightarrow 0$$

One should be careful: this is (generally) not the same as saying all of the states that are marked by a 1 in the complex. Another way of defining this map is by our gluing. Consider the space $G \sqcup \{v'\}$, and let the space $Q = V/xV$. The multiplication map gives an action of V on $Ch^{\bullet}(G)$, via $m_{vv'} : Ch^{\bullet}(G) \otimes V \to G$, where V is the identified with the connected component $\{v'\}$. This means that we can take the tensor product over V

$$\widetilde{Ch}^{\bullet}(G) := Ch^{\bullet}(G) \otimes_V Q$$

One question is whether the vertex chosen for multiplication changes the Reduced chromatic homology theory that we get. (When looking at knots, the answer is no.) Another interesting question is what torsions can show up in the reduced chromatic theory. One might hope that there would be symmetry in two torsion of homology due to a splitting of the homology theory like in [10].

4.2 Counting λ colorings

One can count the number of 2 colorings of a graph by looking at a slightly modified TQFT due to Lee [9]. One question to ask is if you could use an even more modified TQFT to count the number of λ colorings of a graph.

4.3 Other TQFTS

In this section, we described only one TQFT to produce a homology theory. One can show that you can use any R algebra to get a similar construction for the homology theory [6]. However, one might ask what other TQFTs could use to get interesting invariants. The TQFT we used was forgetful in the sense that the only data it could pull out was the number of connected components in the graph. We also have TQFTs that capture too much data. For instance, $C^i(G)$ is a TQFT for every i, and $C^1(G)$ contains enough information to reconstruct the entire graph. Are there less-forgetful TQFTs that give us some interesting topological data of the graph—for instance, functors that capture the graph up to contracting edges that have a degree 2 endpoint? Do these functors give us invariants that are strictly stronger than the chromatic polynomial even when forgetting data like torsion?

4.4 Discrete Morse Theory and b-Cobordisms

In Khovanov Homology, knowing topological data about the cobordisms that link knots together can be used to retrieve data on the knots. One nice thing about the elementary decomposition of b-cobordisms is that they should easy to apply discrete morse theory on. How does the number of critical points in a b-cobordism effect the topology of the graph. Conversely, does information about the homology bound what type of b-cobordisms there can be between two graphs. Are there things like the Rasmussen s-invariant for graphs, and what do they mean?

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