1 Why is it called Mirror Symmetry?

Mirror symmetry is a proposed duality between symplectic geometry and complex geometry. In its full generality, the Mirror symmetry conjecture gives us two things:

- For every symplectic manifold X, a complex manifold $\check{X}$ called the *mirror of* X (and vice versa.)
- A dictionary for translating the symplectic invariants of X into complex invariants on $\check{X}$.
- Some influential papers on mirror symmetry include [CXGP91] [Kon94], [SYZ96] and [GS03].

We still have a lot to learn about both parts of this process, but I want to touch upon some of the earlier results and conjectures of mirror symmetry, and how those led to some modern formulations of the mirror symmetry.
Let's introduce the central player for our version of Mirror Symmetry.

Definition 1 (Calabi-Yau)**.** *A* Calabi-Yau *manifold is a manifold X equipped with the following additional structures:*

- *A holomorphic complex structure $J : TX \to TX$.*
- *A symplectic structure $\omega \in \Omega^2(X)$.*
- *The resulting Kähler metric $g(v, w) := \omega(Jv, w)$ is* Ricci flat.

Historically, physicist have be interested in Calabi-Yau manifolds for their relation to superstring theory. These Calabi-Yau manifolds are a candidate for the "small twisted up additional 6 dimensions" of space-time which we cannot see. It was first believed that there were a small number of Calabi-Yau 3-folds (up to deformation,) which was good as each Calabi-Yau 3-fold gives us a different version of superstring theory. This started a search for different possible string geometries which turned up many more Calabi-Yaus than expected– see for example [Hüb92]
Mathematically, Calabi-Yaus represent the intersection of symplectic and complex geometry.

Theorem 1 (Calabi)**.** *The following are equivalent:*

- *X is Calabi-Yau.*
- *X is Kähler and has a holomorphic volume form.*
- *X is Kähler and has trivial canonical bundle ($\Omega^n \simeq \mathbb{C}$.)*
- *The monodromy of the Kahler metric is contained in $SU(n)$.*

Yau proved this theorem in [Yau78], and (morally) it means Calabi condition is exactly the condition that means the intersection of symplectic, complex and Riemann geometry.

1.1 Hodge Diamond and Deformation

The first invariants for studying Calabi-Yaus come from the complex geometry; it wasn't until much later that we had symplectic tools for understanding the geometry of Calabi-Yaus. Here is a quick review of Hodge theory for Kähler manifolds.

- A complex manifold has a decomposition $\Omega^n(X) \otimes \mathbb{C}$ into $\Omega^{p,q}(X)$ of complex (p, q)-forms. Since the metric is compatible with the complex structure, it makes sense to ask to look at the space of harmonic (p, q)-forms, which we will denote

$$H^{p,q}(X) := \{\omega \in \Omega^{p,q} \mid \Delta\omega = 0.\}$$

- The fundamental result of Hodge theory is that these can be related back to the Dolbeaut cohomology by
$$H^p(X, \Omega^q) = H^{p,q}(X).$$

This is a kind of bigraded version of homology; and we store the numbers $h^{p,q} = \dim H^{p,q}(X)$ in the data of the *Hodge Diamond* for X.

Fact 1. *Here are some facts about the Hodge diamond:*

- *We can recover the original homology of X by*
$$H^n(X) = \bigoplus_{p+q=n} H^{p,q}(X).$$
- *The Hodge star gives us a duality between*
$$H^{p,q} = H^{n-p,n-q}.$$
- *If X is Calabi-Yau, we have another symmetry:*
$$H^{p,0} = H^{n-p,n} = H^{n-p}(X, \Omega^n) = H^{n-p}(X, \mathcal{O}) = H^{n-p,0}.$$

The Hodge diamond has a lot of symmetry! We're now going to make the additional assumption that $h^{1,0} = 0$. Then the Hodge diamond of a Calabi-Yau 3-fold looks like

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Notice that after incorporating all of these symmetries, the Hodge diamond boils down to two numbers $h^{1,1}$ and $h^{2,1}$. What do these two numbers geometrically mean?

1.1.1 $h^{1,1}$, and the Kähler Cone

I think of the two of these guys, $h^{1,1}$ is the one that is easier to get a grasp on– it's related to deformations of the symplectic structure which are compatible with the complex structure on X. A deformation of a symplectic form can be associated with an element of $\Omega^2(X)$.

- For the deformation to be compatible with the complex structure chosen, we have that this elements $\eta \in \Omega^{1,1}(X)$.
- Since the resulting deformation should be again a closed form, we have that $\bar{\partial}\eta = 0$.
- Finally, some deformations of the symplectic form which we can get by a flow. The flow being a symplectomorphism corresponds to the dual one form being closed; the deformations that these flows give are the image of the map $\bar{\partial} : H^{0,1}(X) \to H^{1,1}(X)$.
- Therefore, we can identify the space of deformations with
$$\frac{\ker(\bar{\partial} : \Omega^{1,1}(X) \to \Omega^{1,2}(X))}{\text{Im } (\bar{\partial} : \Omega^{0,1}(X) \to \Omega^{1,1}(X).)}$$

The big take away is

> $h^{1,1}$ tells us about deformations of symplectic structure.

1.1.2 $h^{2,1}$, and complex deformation

What about deformations of complex structure? Let's look at two almost structures, J and J'. Then we can look at the image of the holomorphic tangent bundle of J with respect to complex multiplication by J'.

$$J(T_J(X)^{1,0}) \subset T(X)^{1,0} \otimes \mathbb{C} \simeq T_J(X)^{0,1} \oplus T_J^{0,1}(X).$$

In good circumstances, this is the graph of a function $A : T_J X^{1,0} \to T_J X^{0,1}$, which is the same as an element of $s \in \Omega^{0,1}(X, TX^{1,0})$.

- One can check that the condition that this new structure to be integrable is equivalent to $\bar{\partial} s = 0$.
- There are some deformations of complex structure that arise from pullback of a diffeomorphism; these are given by the image of $\bar{\partial} : \Omega^{0,0}(X, TX^{1,0}) \to \Omega^1(X, TX^{1,0})$.
- The space of deformations of complex structure is given by

$$\frac{\ker(\bar{\partial} : \Omega^{0,1}(X, TX^{1,0}) \to \Omega^{0,2}(X, TX^{1,0})}{\text{Im } (\bar{\partial} : \Omega^{0,0}(X, TX^{1,0}) \to \Omega^{0,1}(X, TX^{1,0})} = H^{0,1}(X, TX^{1,0})$$

  On a Calabi-Yau manifold, we have

$$H^1(X, TX^{1,0}) = H^1(X, \Omega^{n-1,1}) = H^{1,n-1}(X) = H^{1,2}(X) = H^{2,1}(X).$$

This map is called the *Kodaira Spencer map,* and on Calabi-Yau manifolds this first order approximation to deformations of complex structure is unobstructed. So, the big take away is

> $h^{2,1}$ tells us about deformations of symplectic structure.

1.2 Why is it called Mirror Symmetry

At last, we have gotten to the origin of the name "Mirror Symmetry." If X and $\check{X}$ are Mirror symmetric, then variations of symplectic structure on X should correspond to variations of complex structure on $\check{X}$ and vice versa. Since the dimensions of these moduli space are governed by the Hodge numbers, we get that $h^{1,1}(X) = h^{2,1}(\check{X})$ and vice versa. Pictorially the Hodge diamonds of X and $\check{X}$ are related by a mirror flip along the diagonal.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1}(X) & & 0 & \\ 1 & & h^{2,1}(X) & & h^{2,1}(X) & & 1 \\ & 0 & & h^{1,1}(X) & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,1}(\check{X}) & & 0 & \\ 1 & & h^{1,1}(\check{X}) & & h^{1,1}(\check{X}) & & 1 \\ & 0 & & h^{2,1}(\check{X}) & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

During the search for new string geometries, many mirror pairs of Calabi-Yaus were found providing a lot of empirical evidence for the existence for the Mirror Symmetry conjecture.

2 Constructing Mirror Manifolds, and Implications

We've now seen why we call this duality Mirror symmetry, but I'd like to outline [1] how one might go about constructing mirror manifolds (and why we might expect it to be true,) and the implications of mirror symmetry (including the excitement that got this field kicked off in the 1990's.)

2.1 Constructing Mirrors

The first examples of Mirror symmetry were for Calabi-Yau hypersurfaces of projective space. We can compute the Chern class of the hypersurface by the exact sequence:

$$0 \to T^{1,0}(H) \to T^{1,0}\mathbb{CP}^n \to N_H \to 0$$

This tells us that $c(\mathbb{CP}^n)|_H = c(H) \cdot c(N_H)$.
Suppose that the hypersurface is degree d. Then we can compute

$$c(\mathbb{CP}^n) = (1+\omega)^{n+1}$$

$$c(N_H) = (1+\omega)^d$$

and therefore

$$1 + (n+1)\omega + \cdots = 1 + d\omega + c_1(H) + \cdots$$

so that

$$c_1(H) = (n+1-d)\omega.$$

Notice is $d = n+1$, is a necessary condition for our manifold to be Calabi-Yau; with a bit more work one can show that it is a sufficient condition.
In this case, there is a particularly degenerate deformation of X to the union of $\mathbb{CP}^{n-1}$ hyperplanes, called the *large complex structure limit.* In this deformation limit, X is a toric manifold. A general strategy for constructing mirrors might to do mirror symmetry in families. Let F be a family of Calabi-Yau manifolds so that X_0 is toric (and possibly degenerate.) Then our strategy will be

1. Find a degeneration of F to a toric variety X_0
2. Find a mirror $\check{X}_0$ for X_0
3. Match the deformation parameters for $\check{F}$ and $\check{X}$.

There are several motivating reasons for taking this path:

- Physicists propose that Mirror Symmetry is T-duality, which says that we should look at dual Tori.
- The homological mirror symmetry conjecture posits that Lagrangian tori should correspond to skyscraper sheaves on the mirror, and so torus fibrations are good things to look at.
- The topological reasons below:

Here is also a motivating calculation from [Gro01]– (ignoring many details about the singular fibers, which need to be incorporated for this to make sense.) Suppose that we have a torus fibration $f : X \to B$ and suppose that X is simply connected[2]. Let's additionally assume that the fibers of this map are special lagrangian. This means that the real holomorphic volume form is a volume form on the fiber $F = f^{-1}(p)$, so there is a natural isomorphism

$$\begin{tikzcd} H^3(F) \arrow[rr] \arrow[dr] & & \mathbb{C} \arrow[dl] \\ & B & \end{tikzcd}$$

as bundles over B.

[1] and I really mean outline
[2] Look; this already doesn't make sense!

- Let's compute $H^3(X, \mathbb{C})$ two different ways.
  On the one hand, Hodge theory tells us that

$$H^2(X, \mathbb{C}) = h^{1,1} = h^{2,2} = H^4(X, \mathbb{C}).$$

$$H^3(X, \mathbb{C}) = h^{1,2} + 2.$$

  On the other hand, we can use the Serre spectral sequence to compute the homology based on the map

$$f : X \to B.$$

  Claim 1. *The second page of the Serre spectral sequence converges and is isomorphic to the Hodge diamond, that is:*

$$H^p(B, H^q(F)) = H^{p,q}(X),$$

  where $H^q(F)$ is the bundle of the homology of the fiber F.

  The E_2 page of the spectral sequence looks like

$$\begin{array}{cccc}
H^0(B, H^3(F)) & H^1(B, H^3(F)) & H^2(B, H^3(F)) & H^3(B, H^3(F)) \\
H^0(B, H^2(F)) & H^1(B, H^2(F)) & H^2(B, H^2(F)) & H^3(B, H^2(F)) \\
H^0(B, H^1(F)) & H^1(B, H^1(F)) & H^2(B, H^1(F)) & H^3(B, H^1(F)) \\
H^0(B, H^0(F)) & H^1(B, H^0(F)) & H^2(B, H^0(F)) & H^3(B, H^0(F))
\end{array}$$

  - The important thing here is special Lagrangian condition on the fibration gives us a real volume form on the fibers of our Lagrangians. This is an isomorphism between $H^3(F) \to \mathbb{C}$. Also, $H^0(F) = \mathbb{C}$.

$$\begin{array}{cccc}
H^0(B, \mathbb{C}) & H^1(B, \mathbb{C}) & H^2(B, \mathbb{C}) & H^3(B, \mathbb{C}) \\
H^0(B, H^2(F)) & H^1(B, H^2(F)) & H^2(B, H^2(F)) & H^3(B, H^2(F)) \\
H^0(B, H^1(F)) & H^1(B, H^1(F)) & H^2(B, H^1(F)) & H^3(B, H^1(F)) \\
H^0(B, \mathbb{C}) & H^1(B, \mathbb{C}) & H^2(B, \mathbb{C}) & H^3(B, \mathbb{C})
\end{array}$$

  - Since B is simply connected, we have that $H^1(B, \mathbb{C}) = 0$ and $H^2(B, \mathbb{C}) = 0$. We also know the four corners.

$$\begin{array}{cccc}
\mathbb{C} & 0 & 0 & \mathbb{C} \\
H^0(B, H^2(F)) & H^1(B, H^2(F)) & H^2(B, H^2(F)) & H^3(B, H^2(F)) \\
H^0(B, H^1(F)) & H^1(B, H^1(F)) & H^2(B, H^1(F)) & H^3(B, H^1(F)) \\
\mathbb{C} & 0 & 0 & \mathbb{C}
\end{array}$$

- Since we assumed that $H^1(X, \mathbb{C}) = 0$, we have by spectral sequence reasons that $H^0(B, H^1(F) = 0$ and $H^3(B, H^2(F)) = 0$

$$\begin{array}{cccc}
\mathbb{C} & 0 & 0 & \mathbb{C} \\
H^0(B, H^2(F)) & H^1(B, H^2(F)) & H^2(B, H^2(F)) & 0 \\
0 & H^1(B, H^1(F)) & H^2(B, H^1(F)) & H^3(B, H^1(F)) \\
\mathbb{C} & 0 & 0 & \mathbb{C}
\end{array}$$

- Now, let's look at the dual torus fibration, $\check{f} : \check{X} \to B$ with the property that it is isomorphic to

$$\pi : H^1(F)/\mathbb{Z} \to B.$$

Given an element $\alpha \in H^1(F)$, we can pair it with an element $\beta \in H^1(H^1(F)/\mathbb{Z})$ by taking the vector field in the direction of α,

$$\beta(\alpha) = \beta(X_\alpha)$$

which identified $H^1(\check{F}) = H_1(F)$. We can similarly show that $H^k(\check{F}) = H_k(F)$. This is an isomorphism of bundles over B.
Applying the same arguments we made before for $X \to B$, we get have that $H^3(B, H^2(\check{F})) = 0$ and $H^0(B, H^1(\check{F})) = 0$. Additionally, Poincaré duality tells us that

$$H^2(F) = H_2(F) = H^1(\check{F})$$

So $H^0(B, H^2(F)) = H^0(B, \check{F}) = 0$.

$$\begin{array}{cccc}
\mathbb{C} & 0 & 0 & \mathbb{C} \\
0 & H^1(B, H^2(F)) & H^2(B, H^2(F)) & 0 \\
0 & H^1(B, H^1(F)) & H^2(B, H^1(F)) & 0 \\
\mathbb{C} & 0 & 0 & \mathbb{C}
\end{array}$$

- One can show that the spectral sequence converges on the E_2 page, so we've finished computing homology.

• Now, what results do we have from this computation of homology?

$$H^1(B, H^1(F)) = H^2(X, \mathbb{C}) = h^1 1(X) = h^2 2(X) = H^4(X, \mathbb{C}) = H^2(B, H^1(F))$$

And on the mirror

$$H^1(B, H^1(\check{F})) = H^2(\check{X}, \mathbb{C}) = h^1 1 = h^2 2 = H^4(\check{X}, \mathbb{C}) = H^2(B, H^1(\check{F}))$$

Notice that $H^1(B, H^1(\check{F})) = H^1(B, H^2(F))$. So, we get the following (previously unknown) equalities:

$$H^1(B, H^2(F)) = H^1(B, H^1(\check{F})) = H^2(B, H^2(\check{F}) H^2(B, H^1(F))$$

Since $H^3(X,\mathbb{C}) = \mathbb{C}\oplus H^1(B,H^2(F))\oplus H^2(B,H^1(F))\oplus\mathbb{C}$, this last inequality tells us that $\dim(H^3(X,\mathbb{C})) = 2\dim(H^1(B,H^2(F)) + 2$ and therefore $h^{1,2}(X) = \dim(H^1(B,H^2(F))$. Our mirror statement is then

$$h^{1,2}(X) = \dim(H^1(B,H^2(F)) = \dim(H^1(B,H^1(\check{F})) = h^{1,1}(\check{X}).$$

$$h^{1,1}(X) = \dim(H^1(B,H^1(F)) = \dim(H^1(B,H^2(\check{F})) = h^{1,2}(\check{X}).$$

3 Where to go from here?

If mirror symmetry tells us that deformations of symplectic structures should relate to deformations of complex structures, then symplectic/complex invariants of spaces should have a similar deformation theory. One of the original predictions of mirror symmetry was a pairing between two different couplings of the theory:

- *The symplectic Yukawa coupling.* Pick $\alpha_1, \alpha_2, \alpha_3 \in H^{1,1}(X,\mathbb{C})$. Let ω be the symplectic form, and $\beta \in H^2(X,\mathbb{C})$. Define the 3-pointed Gromov witten invariants as:

  $$\langle \alpha_1, \alpha_2, \alpha_3\rangle_\beta := n_\beta \int_\beta \alpha_1 \int_\beta \alpha_2 \int_\beta \alpha_3$$

  where n_β is the number of holomorphic spheres in the class of β. If the α_i are dual to submanifolds a_1, a_2, a_3, this is counting the number of holomorphic spheres which are in the class of β and intersect a_1, a_2 and a_3. We can remove the dependence on β by packaging this into the Yukawa Coupling (only dependent on ω)

  $$\langle \alpha_1, \alpha_2, \alpha_3\rangle = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \sum_\beta \langle \alpha_1, \alpha_2, \alpha_3\rangle_\beta e^{-2\pi \int_\beta \omega}$$

  Notice that the Yukawa coupling is given as power series expansion in β, with coefficients given by the 3 point Gromov-Witten invariants. In this sense, we can think of the Yukawa coupling as being the intersection triple product to first order, and having higher contribution terms coming in from contributions of ω.
  When we take deformations of the symplectic structure, the Yukawa coupling deforms as well. The deformation theory is given by the higher order contributions, which are the Gromov-Witten invariants.

- *The complex Yukawa coupling.* Pick $a_1, a_2, a_3 \in H^1(\check{X}, T\check{X}) = H^{2,1}(\check{X})$. Then there is a triple coupling:

  $$\langle a_1, a_2, a_3\rangle = \int_{\check{X}} \Omega \wedge (\Omega \otimes a_1 \otimes a_2 \otimes a_3)$$

  Taking this and expanding it out as a power series in terms of power series in the deformations of Ω will give us another generating function.
  Although we have written this out nicely as a product, we could also write it out as a power series in H^2.

These are both deformations of the triple product structure on $H^2(X)$, and mirror symmetry says that there are coordinates on $H^{1,1}(X)$ and $H^{1,2}(\check{X})$ so that the power series expansions are the same. This is the original version of mirror symmetry as stated in [CXGP91].

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