An exposition on the Fukaya-Morse Algebra

Jeff Hicks

August 1, 2017

Abstract

In this paper we give a new proof of invariance of the Morse-Fukaya algebra.

Morse theory starts with a manifold L with a smooth function $H : L \to \mathbb{R}$, and returns a cohomology theory $(C^\bullet(L; H), d)$ where $C^k(L; H)$ is the $\mathbb{R}$-vector space with basis given by critical points of f, and d count gradient flow lines between those critical points. It's a beautiful construction, drawing in pieces of geometry, topology, analysis and homological algebra together to provide a picture-friendly interpretation to cohomology. Two wonderful expositions of the subject are [AD13] and [Hut02], which provide an detailed construction and applications of Morse Cohomology. We'll assume some familiarity with the construction of Morse theory, and fix some notation at the start.

Morse cohomology takes as it's input data a triple (L, g, H), where L is a manifold, $g : TL \times TL \to \mathbb{R}$ is a metric, and $H : L \to \mathbb{R}$ is a smooth function. Not any triple will do; we ask for the following conditions to hold:

- The function H should be *Morse*, so that it has isolated critical points and the Hessian at each critical point is non-degenerate. The number of positive eigenvalues of the Hessian is called the *index* of the critical point, and will be denoted $\operatorname{ind}(f)$.
- Our choice of Morse function H gives us a map $\phi_t : L \times \mathbb{R} \to L$ given by the flow of the gradient vector field on X. To each critical point $p \in \operatorname{Crit}(f)$, we can associate a *upward and downward* flow manifold, given by
$$W_p^+ := \{x \in X \mid \lim_{t\to\infty}(x) = p\}$$
$$W_p^- := \{x \in X \mid \lim_{t\to-\infty}(x) = p\}$$
The dimension of the upward manifold is $\dim W_p^+ = \operatorname{ind} p$, and the downward manifold has dimension $\dim W_p^- = \dim X - \operatorname{ind} p$. We require that $W_p^+ \cap W_q^-$ is a transverse interaction for every pair of critical points p and q.

From this data, we can associate to every pair of critical points p and q the *moduli space of flow lines* between p and q,
$$\mathcal{M}_1(p, q) := (W_p^- \cap W_q^+)/\mathbb{R}$$
The $\mathbb{R}$ action on $M_q(p, q)$ comes from ϕ_t. By construction, whenever this manifold is non-empty, it's dimension $\dim(\mathcal{M}_1(p, q)) = \operatorname{ind}(q) - \operatorname{ind}(p) - 1$. This moduli space is a smooth, not-necessarily compact manifold. However, it admits a compactification $\bar{\mathcal{M}}_\infty(p, q)$ whose codimension 1 boundary strata counts *broken flow lines:*
$$\partial\bar{\mathcal{M}}_\infty(p, r) = \bigsqcup_{p<q<r} \mathcal{M}_1(p, q) \times \mathcal{M}_1(q, r).$$
The *Morse Complex*[1] $C(L; H)$ is a chain complex with coefficients in $\mathbb{R}$, which as a vector space whose basis is the critical points of f,
$$C^k(L; H) := \mathbb{R}\langle p \in \operatorname{Crit}(f), \operatorname{ind} p = k\rangle$$

[1] Here, we suppress the g.

The differential on this complex is given by counting flow lines

$$\langle d(p), q\rangle := \#\mathcal{M}(p,q).$$

The boundary compactification by broken flow lines is used to prove the relation $d^2 = 0$.
Having fixed some notation, we are ready to summarize the remainder of this paper.

- The first goal is to provide further exposition on the Morse-Fukaya algebra as described in [Fuk96]. When one tries to give $C^\bullet(L, H)$ a product structure, one is forced to consider the structure of A_∞ algebra. We'll provide a picture-heavy exploration of the Fukaya-Morse algebra, giving motivation for A_∞ structures.
- The second section examine how we show that the resulting complex is independent of the choice of Morse function. Our proof is very geometric, and an extension of the classical proof of invariance of the Morse complex of L. In the classical proof one can construct a chain map between $C(L; H_0)$ and $C(L; H_t)$ by considering an interpolating function $C(L \times I, H_t)$. However, this proof will not yield an A_∞ morphism between these two chain complexes. Instead, we will show that if one wants to construct an A_∞ morphism between these two complexes, one needs to consider an interpolating Morse function on a large parameter space
$$C(L \times I^k, f_{t_1,\ldots,t_k}.)$$

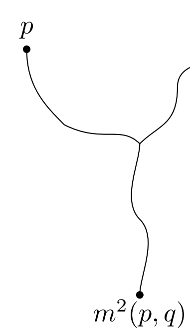
Both of these sections will be presented in three portions:

- An expectation on how we should construct these objects and where that expectation breaks down
- A discussion of the combinatorial objects involved in fixing these expectations
- The resulting moduli space of objects one is forced to consider as a result, and a proof of the resulting algebraic structure.

1 Morse-Fukaya

1.1 First Attempt

How should one take the product of two critical points when given a Morse function to work with. If a flow line gives us a geometric intution on how to construct the differential, one might reasonably hope that a *flow tree* captures the requisite information for constructing a product operation on the space of critical points.



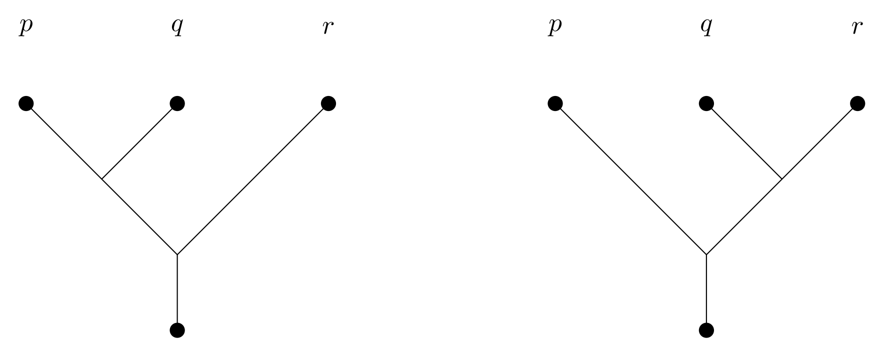
Unfortunately, this hope runs into two problems:

- A obvious problem comes from the fact that if all 3 flow lines in a flow tree are flow lines of grad H, then they must all be the same flow line. There are two different ways to solve this problem: one can either work with a variety of different Morse functions, or one can choose small *perturbations* to the Morse flow in order to make the intersections of flow spaces corresponding to the product transverse.

$$PerturbationProblem$$

We will outline some of the considerations needed to take the perturbative approach in Section ??. If one instead considers multiple different Morse functions, one would construct the Fukaya-Morse category.

- Outside of the analytic difficulties in constructing a product, we also have interesting geometry that pops up when we try to prove algebraic properties of the product. One may hope, for instance, that this product is associative. Unfortunately, there is no expectation for the product to be associative, as the different orders of composition give us very different looking trees.



$m^2(m^2(p, q), r)$ $m^2(p, m^2(q, r))$

This is not surprising, considering that one should only hope for product operations of cohomology to be associative up to a homotopy. Where Morse theory differs from cohomology is that the Morse function explicitly gives us the data we need to construct that homotopy, by considering trees with three inputs. This story repeats itself, as the choice of homotopy data for associativity is itself only defined up to an explicit homotopy by studying trees with 4 inputs. This discussion naturally leads us to considering the combinatorial construction of the Stasheff associahedra, its relation to metric trees, and the A_∞ -algebra.

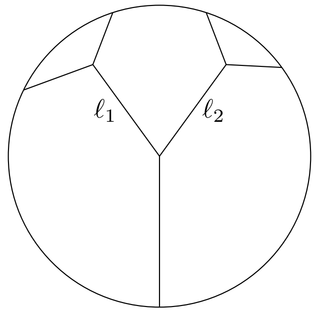
1.2 Combinatorial Objects

The homotopies and higher homotopies of product structure that we previewed just a few sentences ago are artifacts of the combinatorics of metric trees.

Definition 1. *A* ribbon metric tree (T, ℓ) *is a tree* $T = (V, E)$ *with the following additional data:*

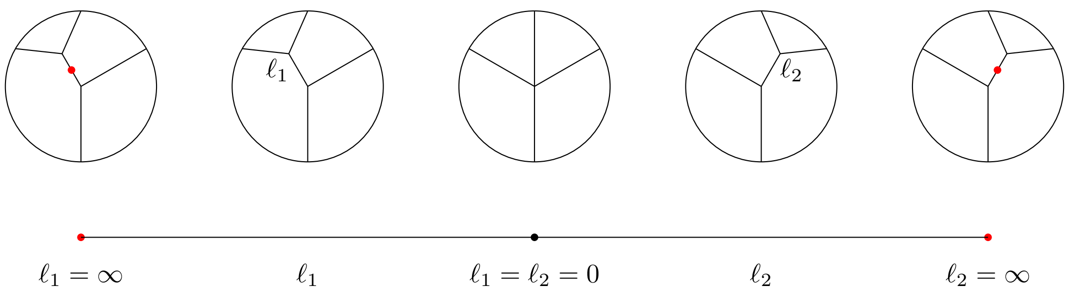
- *A ordering of the leaves of* T*, and a selection of a root.*
- *An assignment of lengths in* $[0, \infty]$ *to each of the edges of* T*, where each internal edge has finite length, and each external edge receives length* ∞*.*

The valency $\nu(T)$ *of a metric tree is the number of leaves.*
The combinatorial type of a metric tree is the tree T*.*



The metric trees of fixed combinatorial type T can be identified with the space $\mathcal{M}_T := [0,\infty)^{|E|-\nu(T)}$. One expectation that we get from the pictures is that the boundaries of $\mathcal{M}_T$ coming from length zero edges correspond to metric trees where various edges have been contracted.

Claim 1 (The contraction identification)**.** *Let T/e be obtained by T by contracting an edge e. Then there is a map, bijective onto the image, from $\mathcal{M}_{T/e} \times [-,\epsilon) \to \mathcal{M}_T$.*

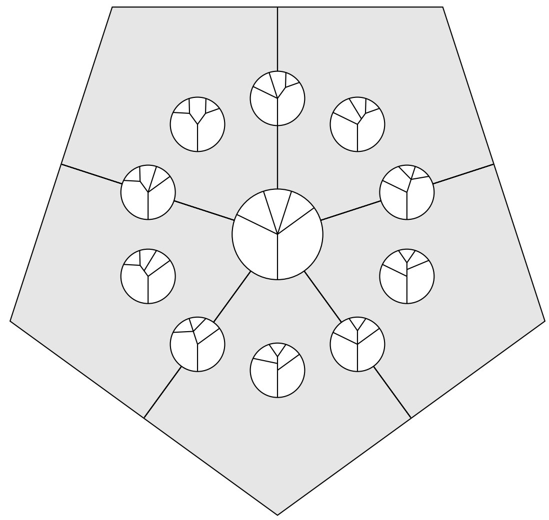


Without considering contraction, the space of all valence k-trees is a disjoint set; however, after making the natural identification of length 0 edges with contracted edges, each space $\mathcal{M}_T$ with $\nu(T) = k$ becomes a cell in a CW-complex

$$\mathcal{M}_k := \bigcup_{\nu(T)=k} M_T$$

called the *Stasheff associahedra.*

Theorem 1. *The space of ribbon metric trees with valency k, $\mathcal{M}_k$ is a smooth manifold.*

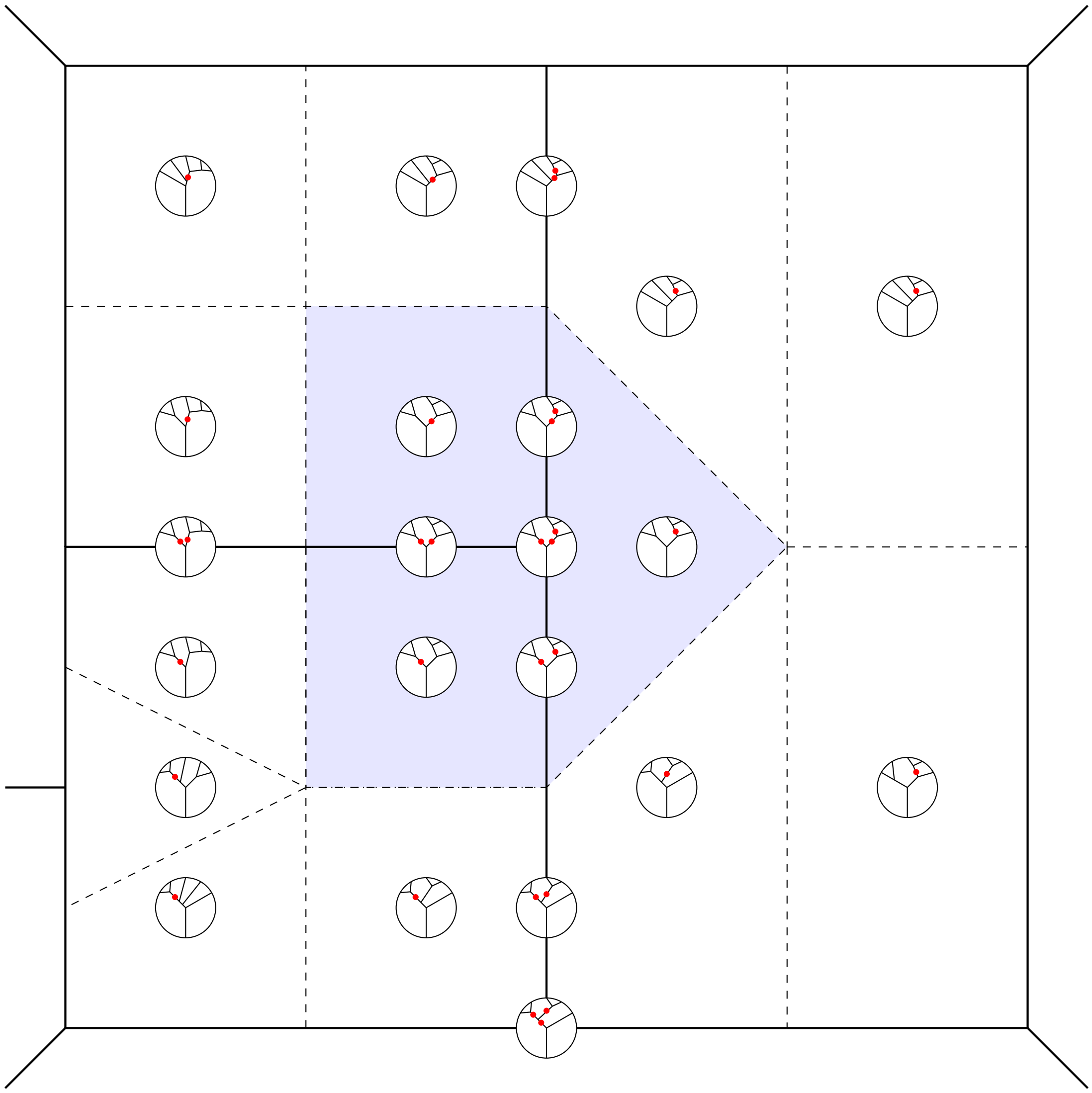


The space of ribbon metric trees with valency k is noncompact as edge may have lengths that diverge to infinity. However, this non-compactness can be understood as a "breaking" of the ribbon tree into two smaller ribbon trees along the unbounded edge.

Theorem 2. *There is a compactification $\bar{\mathcal{M}}_{||}$ into a manifold with corners. The k-codimensional strata are given by k-broken ribbon trees. In particular, the one dimensional strata can be described as*

$$\bigcup_{1\leq i\leq j\leq k} \mathcal{M}_{j-i+1} \times \mathcal{M}_{k-j+i}$$

While we skip a proof of this theorem, we'll provide a pretty picture showing a portion boundary of $\mathcal{M}_5$, where on sees pentagonal and square facets, corresponding to the boundary components $\mathcal{M}_2 \times \mathcal{M}_4$ and $\mathcal{M}_3 \times \mathcal{M}_3$ respectively. Here is a drawing of three of the facets of the associahedra, two pentagons and a square. The cell decomposition is given by the dotted lines, and I've highlighted a cell in light blue. One point of confusion about the associahedra: the cell decomposition does not give us the boundary stratification; rather we identify each cell with $[0, \infty]^3$, and the "infinite" boundary of these cells each contribute a portion of the boundary stratification of $\mathcal{M}_5$. Notice that each cell is a hexagon with a trivalent vertex on the inside. This corresponds to the 6 edges of $[0, \infty]^3$ which have a coordinate set to ∞. Each cell corresponds to a tree of fixed combinatorial type, while each face corresponds to several types of trees exhibiting the same breaking property. I've additionally labeled some of the codimension 2 and 3 boundary strata of the associahedra, which correspond to trees that have been broken 2 and 3 times respectively.



1.3 The Morse Theory:Flow Trees

When we are given (L, H), a manifold equipped with a Morse function, we can encode the combinatorics of ribbon metric trees into the Morse theory of (L, H) by replacing the edges of the ribbon metric tree with flow lines of H. The combinatorics of the ribbon tree will tell us about the incidence relations of the flow lines, while the lengths of the edges on the ribbon tree will tell us how long we flow along each of the curves in the tree. Finally, the boundary strata of broken ribbon trees will tell us about the boundary of the moduli space of flow trees.

Let T be the topological realization of a tree. We will study maps $u : T \to M$ where the edges correspond to Morse flow lines of f and whose leaves limit to critical points of the Morse function. If we naïvely take this approach the resulting trees will be entirely contained within a flow line; we will therefore need to add perturbative data to the edges of the tree to achieve the necessary transversality to have an interesting count of flow trees.

Definition 2. *Let (T, ℓ) be a metric tree. A* perturbation datum *for T is a set of choices of smooth* *perturbations for each edge*

$$p_e : [0, \ell(e)] \times M \to \mathbb{R}$$

which limits to zero at the leaves of T.
A collection of perturbation datum for every metric tree will be called a perturbation datum for (M, H) and denoted $\mathcal{P}$. A perturbation datum $\mathcal{P}$ is coherent *if $p_{(T,\ell)}$ depends smoothly on (T, ℓ) in the moduli space of trees of valency $\nu(T)$, and the perturbation datum on broken trees in the stratum of $\bar{\mathcal{M}}_{||}$ matches the of the perturbation datum on $\mathcal{M}_i \times \mathcal{M}_j$ after identifying broken trees.*
A perturbation datum for T is transverse *if for every vertex v with incident edges $e_1, e_2, \ldots, e_k$, the Morse functions $H + p_{e_i}$ have transverse gradients away from the critical points.*

A non-trivial analytic result is to show that coherent choices of perturbations exist.

Theorem 3 ([Abo09; Mes16])**.** *There exist a choice of coherent transverse perturbation datum.*

This perturbation set up makes the count of trees whose edges match gradient flow lines reasonable.

Definition 3. *A labeling of a metric tree is an assignment to each of the leaves e_k a critical point c_i.*
Let $(\mathcal{T}, \ell)$ be a labeled metric tree, and let $\mathcal{P}$ be a coherent perturbation datum. A Morse $(\mathcal{T}, \ell)$ flow metric tree is a map $u : \mathcal{T} \to G$ which is smooth on edges, and continuous at vertices. This map should satisfy the perturbed gradient flow equation at each edge,

$$\partial_t(u)|_{e_i} = -\operatorname{grad}(H + p_{e_i})$$

with the additional requirement that at each leaf e_i,

$$\lim_{t_{e_i} \to \infty} u(t_{e_i}) = c_i$$

and at the root e_0, we have

$$\lim_{t_{e_0} \to -\infty} u(t_{e_0}) = c_0.$$

The space of all such flow trees in a fixed combinatorial type is denoted

$$\mathcal{M}_T(c_0; c_1, \ldots c_k).$$

The space of the space of Morse flow trees enjoys many of the same properties as the space of Morse flow lines, which may be treated as the moduli space of flow trees with 1-input.

Claim 2. *The space $\mathcal{M}_T(c_0; c_1, \ldots c_k)$ is a smooth noncompact- manifold with corners of dimension*

$$(k-2) - \operatorname{ind}(c_0) + \sum_{i \geq 1} \operatorname{ind}(c_i).$$

The corners of $\mathcal{M}_T(c_0; c_1, \ldots, c_k)$ comes from the flow trees which include an edge of length zero.

Proof. The proof of smoothness is similar to the proof of smoothness of the Morse moduli space, with the additional twist of transversality required. For the proof of dimension, we break into two cases

- If $k = 1$, then we are in the case of Morse flow lines, and we already know the dimension of this moduli space. Notice that the dimension of the moduli space is given by $\dim) W^+_{c_0} \cap W^-_{c_1}) - 1$, where the -1 comes from the fact that we quotient out by the action of $\mathbb{R}$ on $\mathcal{M}_1$.

- If $k \geq 2$, then we can compute

$$\begin{aligned}\dim(\mathcal{M}(c_0; c_1, \dots, c_k) &= \dim(W^+_{c_0} \cap W^-_{c_1} \cap \dots \cap W^-_{c_k}) + \dim(\mathcal{M}_k) \\ &= -\operatorname{ind} c_0 + \dim(\mathcal{M}_k) + \sum_{i=1}^k \operatorname{ind} c_i \\ &= -\operatorname{ind} c_0 + (k-2) + \sum_{i=1}^k \operatorname{ind} c_i\end{aligned}$$

This $(k-2)$ factor is why we treat the case of $k = 1$ separately, as the space of trees where every internal vertex has valency 3 has no non-trivial self-automorphisms.

□

Just like in the Morse setting, these spaces admit compactifications by including broken flow trees. For each $1 \leq i < j \leq k$, define T_{ij} to be the smallest subtree of T which contains the leaves labeled $c_i, \dots, c_j$; similarly, let $T_{k \setminus ij}$ to the subtree $T \setminus T_{ij} \cup e$, where e is the edge connected to the root of T_{ij}.

Claim 3. *The space $\mathcal{M}_T(c_0; c_1, \dots, c_k)$ admits a compactification by including broken flow trees, so that*

$$\mathcal{M}_T(c_0; c_1, \dots, c_k) = \bigcup_{\substack{1 \leq i < j < \leq k \\ d \in \operatorname{Crit}(H)}} \mathcal{M}_{T_{k \setminus ij}}(c_0; c_1, \dots, c_{i-1}, d, c_{j+1}, \dots, c_k) \times \mathcal{M}_{T_{ij}}(d; c_i \dots, c_j).$$

Notice that this decomposition is exactly the same as the boundary stratification of ribbon metric trees. In the setting of flow trees, we have a identification of flow trees that we previously did not see in the Morse case given by the contraction identifications that we had for the space of metric trees.

Claim 4. *Suppose that T/e is obtained from T by contraction along an internal edge. Then there exists an embedding $\mathcal{M}_{T/e}(c_0; c_1, \dots c_k) \times [0, \epsilon) \to \mathcal{M}_T(c_0; c_1, \dots, c_k)$. This embedding is compatible with the boundary compactifications.*

This structure result on the space of metric trees of fixed combinatorial type, and the identification claim allow us to assemble the Morse flow trees of a fixed valency into a smooth manifold. This gives us the major theorem that we need for studying flow trees.

Theorem 4 (Fukaya)**.** *The space of metric trees of combinatorial type T with labels $c_0, \dots, c_k$ is a smooth manifold of dimension*

$$(2-k) + \operatorname{ind}(c_0) - \sum_{i \geq 1} \operatorname{ind}(c_i)$$

Furthermore, there is a compactification of $\mathcal{M}(c_0; c_1, \dots, c_k)$ by broken Morse flow trees with codimension 1 strata given by

$$\partial \bar{\mathcal{M}}(c_0; c_1, \dots, c_k) = \bigcup_{1 \leq i \leq j \leq k} \mathcal{M}(c_0; c_1, \dots, c_{i-1}, d, c_{j+1}, \dots c_k) \times \mathcal{M}(d; c_i, \dots, c_j)$$

Remark 1. *A word of caution: we are sweeping a lot under the rug here. For the purposes of exposition, we've treated the analytic problem of attaching coherent transverse perturbation datum first and the combinatorial problem of handling how the different types of trees glue together second, and not being particularly careful how these problems influence eachother. Also, we've taken our perturbation datum from the Morse functions themselves. An alternate approach that we could have taken would be to assemble the set of maps from $(T, \ell) \to L$ into a* polyfold, *and use abstract perturbative methods to achieve the necessary transversality instead.*

Remark 2. *The choice of index is designed to make the product graded, and makes this a Morse Cohomology. The moduli space $\mathcal{M}(c_0; c_1, \dots, c_k)$ is zero dimensional only when $\operatorname{ind}(c_0) = (2-k) + \operatorname{ind}(c_i)$. If we focus on the trees that are suppose to contribute to our product (which are those trees with $k = 2$), we get a non-trivial contribution to the product when*

$$\operatorname{ind}(c_0) = \operatorname{ind} c_1 + \operatorname{ind} c_2$$

making the induced product by counting these trees graded in index.

1.3.1 Some Notation

I think at this point it becomes a good idea to introduce some additional notation, based on multiindices. Since almost everything we will do here is dependent on order, when we write an index set I we will assume that it is ordered. When we want to write a partition of a set K, we will write $(I_1|I_2|\cdots|I_l) = K$, and always understand that $I_1 < I_2 < \ldots < I_l$.
When we write c_K, we will mean the tuple $(c_1, \dots, c_k)$.

$$\partial\bar{\mathcal{M}}(c_0; c_K) = \bigcup_{(I_1|J|I_2)=K} \mathcal{M}(c_0; c_{I_1}, d, c_{I_2}) \times \mathcal{M}(d; c_J)$$

This notation may take a little bit to get used to, but believe me, it's going to be so worth it in a bit.

1.4 The Corresponding Algebra

There is a lot of data that we've tied together in our construction of the Morse-Fukaya algebra, and we would like an algebraic framework that informs us of the relations between all of these trees that we are counting. In the Morse homology setting, one defines a chain complex on the data $C(L;H) := \mathbb{R}\langle \operatorname{Crit}(H)\rangle$ by counting flow lines between the critical points. Whenever $\operatorname{ind} c_1 = \operatorname{ind} c_0 + 1$, we know that the space $\mathcal{M}(c_0; c_1)$ is a compact zero dimensional manifold, and we may therefore count the points in it[2]. We define the differential to be

$$\langle m^1(c_1), c_0\rangle := \#\mathcal{M}(c_0; c_1)$$

The proof that this differential squares to zero uses the structure provided by Theorem **??**, by noticing that

$$\begin{aligned}
\langle m^1(m^1(c_0)), c_1\rangle &= \left\langle m^1\left(\sum_d \langle (m^1(c_0), d\right), d\rangle d, c_1\right\rangle \\
&= \sum_d \langle m^1(c_0), d)\rangle\langle m^1(d), \rangle c_1\rangle \\
&= \#\mathcal{M}(c_0; d)\#\mathcal{M}(d; c_0) \\
&= \#(\partial\mathcal{M}(c_0; c_1))
\end{aligned}$$

This count must be zero because the oriented count of boundary components of a zero 1 dimensional manifold is zero.
We use this proof as a starting point for understanding the algebraic relations between the higher $\mathcal{M}(c_0; c_1, \dots, c_k)$.

Definition 4. *Let $(L;H)$ be a Morse pair, and suppose we've picked regularizing data so that Theorem* **??** *holds. Then we define the higher Morse products*

$$m^k : C^{\otimes k}(L;H) \to C(L;H)$$

by taking counts of flow trees

$$\langle m^k(c_K), c_0\rangle = \#\mathcal{M}(c_0; c_K).$$

This is a graded map of degree $2-k$.

[2]Technically, we should count these points with orientation. This is a whole discussion that I would like to avoid, but the details of orientations can be found in [Mes16].

This gives us many more interesting products than we may have been interested in initially; for instance, it contains something that looks like the product on cohomology defined by m^2, but it also contains more exotic products m^k as well.
By repeating the proof that $m^1(m^1(c_0)) = 0$ that we used for the Morse differential on these higher products, we get the following (slightly odd) relations.

Theorem 5. *The higher products m^k satisfy the A_∞-*relation*, which is a relation for each k that*

$$0 = \sum_{(I_1|J|I_2)=K} \pm m^{|I_1 \cup I_2|}(c_{I_1}, m^{i+j}(c_J), c_{I_2})$$

These relations are a bit confusing on first glance. I think the easiest way to get a feel for them is to write down the first few of them:

$$m^1(m^1(c_1) = 0$$
$$-m^1(m^2(c_1, c_2)) + m^2(m^1(c_1), c_2) + m^2(c_1, m^1(c_2)) = 0$$
$$-m^1(m^3(c_1, c_2, c_3) + m^2(m^2(c_1, c_2), c_3) - m^2(c_1, m^2(c_2, c_3))$$
$$+m^3(m^1(c_1), c_2, c_3) + m^3(c_1, m^1(c_2), c_3) + m^3(c_1, c_2, m^1(c_3)) = 0$$

The first relation can be interpreted as the matp $m^1 : C(L; H) \to C(L; H)$ giving us a differential. The second relation tells us that the product m^2 and differential m^1 have the Leibniz rule relation. The third relation tells us that the associativity of the product holds up to a homotopy defined by the term m^3.
The name A_∞-relation comes from understanding the associativity rule of the product being loosened up to (possibly infinitely many) homotopy relations. It is not surprising that the terms in the k A_∞ relations exactly correspond to the boundary strata of the k-associahedra.
On first glance this set of relations seems constructed to fit the requirements of having some kind of algebraic structure matching the relations of flow trees. While it seems like A_∞ algebras are forced upon us by the combinatorics of our set up, there are in fact strong algebraic motivations for studying A_∞ relations. While we have not yet developed the notion of A_∞ quasi-isomorphism, they are the natural object to study minimal models of chain complexes.

Theorem 6. *Let (C, m^1_C, m^2_C) be a differential graded algebra. Then there exists an (essentially unique) A_∞ algebra (A, m^k_A) with $m^1_A = 0$ which is A_∞ quasi-isomorphic to (C, m^1).*

2 Invariance

Let's move onto showing that this A_∞ structure that we've looked at is independent of the choice of Morse function used. Let H_0, H_1 be two different Morse functions on L. There are several different ways already existing in the literature to prove that $C(L, H_0)$ is quasi-isomorphic to $C(L, H_1)$. One usually constructs *continuation maps*

$$f^k : C^{\otimes k}(L, H_0) \to C(L, H_1)$$

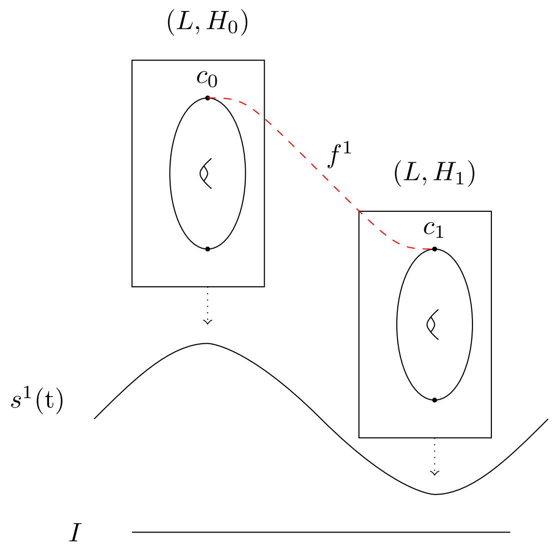
by analyzing flow trees which partially flow along grad H_0, and the continue to flow along grad H_1. This constructs a kind of homotopy between the $m^k_{H_0}$ and $m^k_{H_1}$; by integrating together this homotopy one can construct an A_∞ homomorphism relating $C(L, H_0)$ and $C(L, H_1)$.
The approach that we'll take harkens back to the geometric construction of the invariance of Morse theory.

2.1 A first attempt

The geometric proof of invariance of Morse theory for (L, H) comes from studying Morse theory on $L \times I$. Let H_t be an interpolation between H_0 and H_1, which is constant in t around a small neighborhood of 0 and 1. Let $s^1(t) : I \to \mathbb{R}$ be a S-shaped Morse function as drawn below. We now will study the Morse theory of

$(L \times I, H_t + s^1)$. Provided that we choose s^1 large enough, the critical points of $H_t + s^1$ will be localize to when $t = 0, 1$, so that

$$\mathrm{Crit}(H_t + s^1) = \mathrm{Crit}(H_0) \sqcup \mathrm{Crit}(H_1).$$



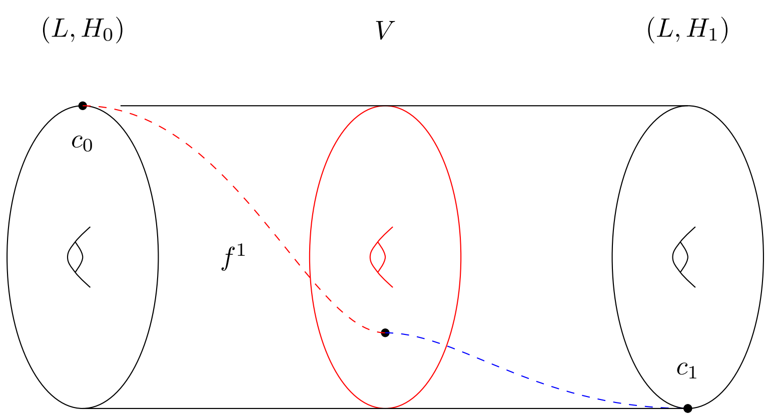
Furthermore, the gradient of $H_t + s^1$ matches the gradient of H_0 and H_1 in a small neighborhood of $t = 0, 1$, so we get that the Morse theory of $L \times I$ can be expressed as

$$C(L \times I, H_t + s^1) = \mathrm{cone}(f^1 : C(L, H_0) \to C(L, H_1))$$

where f^1 is a count of flow lines that go from the critical points above 0 to the critical points above 1. One can geometrically see that the relation $f^1 m_0^1 = m_1^1 f^1$ by noticing that flow lines between $t = 0$ and t_1 must either break on the left or right side; one can also see that this algebraically holds as the coning map must be a chain map.
We would like our geometric argument to stand on slightly firmer ground. To describe the "flow lines that go from the left hand side to the right hand side," we'll introduce a new moduli space. Let V be the hypersurface $t = \frac{1}{2}$ in $L \times I$.

Definition 5. *Let $\mathcal{B}(c_0; c_1)$ be the space of flow lines flowing from c_1 to c_1, with one marked point evaluating to V.*

By construction, half of such a flow line will lie in the $t < 1/2$ portion of $L \times I$, and the other half will lie in the $t > 1/2$ portion, so counting these exactly gives us the flow lines contributing to f^1. Here, the $\mathcal{B}$ stands for *bicolored*, where our coloring comes from whether we have already intercepted $t = 1/2$ or not.



At this point, it's very easy to prove that a count of bicolored flow lines gives us a chain map.

Claim 5. *The space $\mathcal{B}(c_0, c_1)$ is a smooth manifold which admits a compactification by broken flow line $\bar{\mathcal{B}}(c_0, c_1)$. The boundary strata of this compactification is*

$$\partial\bar{\mathcal{B}}(c_0, c_1) = \mathcal{M}_{H_0}(c_0, d) \times \bar{\mathcal{B}}(c; c_1) \sqcup \bar{\mathcal{B}}(c_0, d) \times \mathcal{M}_{H_1}(d, c_1).$$

In short, this claim say that either flow lines break in the red section of a bicolored flow line, or in the blue section. From this lemma, it quickly follows that $f^1 : C(L, H_0) \to C(L, H_1)$ defined by

$$\langle f^1(c_0), c_1\rangle := \#\mathcal{B}(c_0, c_1)$$

is a chain map.
Our goal will now be to extend this argument to construct A_∞ maps between the left and right side. At this point, we run into several complications:

- It's unclear what data an A_∞ homomorphism should carry; certainly it should have more information than just f^1. We'll end up constructing maps $f^k : C^{\otimes k}(L, H_0) \to C(L, H_1)$ coming from bicolored trees, which are the natural generalization of bicolored flow lines. This will be primarily what we explore in 2.2.
- If we wanted to show that the map f^1 was a ring homomorphism, we would have to show something like
$$f^1(m^2_{H^0}(c_1, c_2)) = m^2_{H_1}(f^1(c_1), f^1(c_2)).$$
In order to show this relation, we would look for configurations of flow lines breaking into $f^1(m^2_{H^0}(c_1, c_2))$ and $m^2_{H_1}(f^1(c_1), f^1(c_2)$ in $L \times I$. Unfortunately, the product $m^2_{H_1}$ never occurs in $L \times I$, due to grading reasons. We'll have to find a work-around to this, which will spectacularly spiral out of control as we are forced to accept more and more notation in Section**??** .

2.2 The Combinatorics

The A_∞ homomorphism relations can be geometrically captured through the multiplihedra, which has a natural interpretation as the space of bicolored trees.

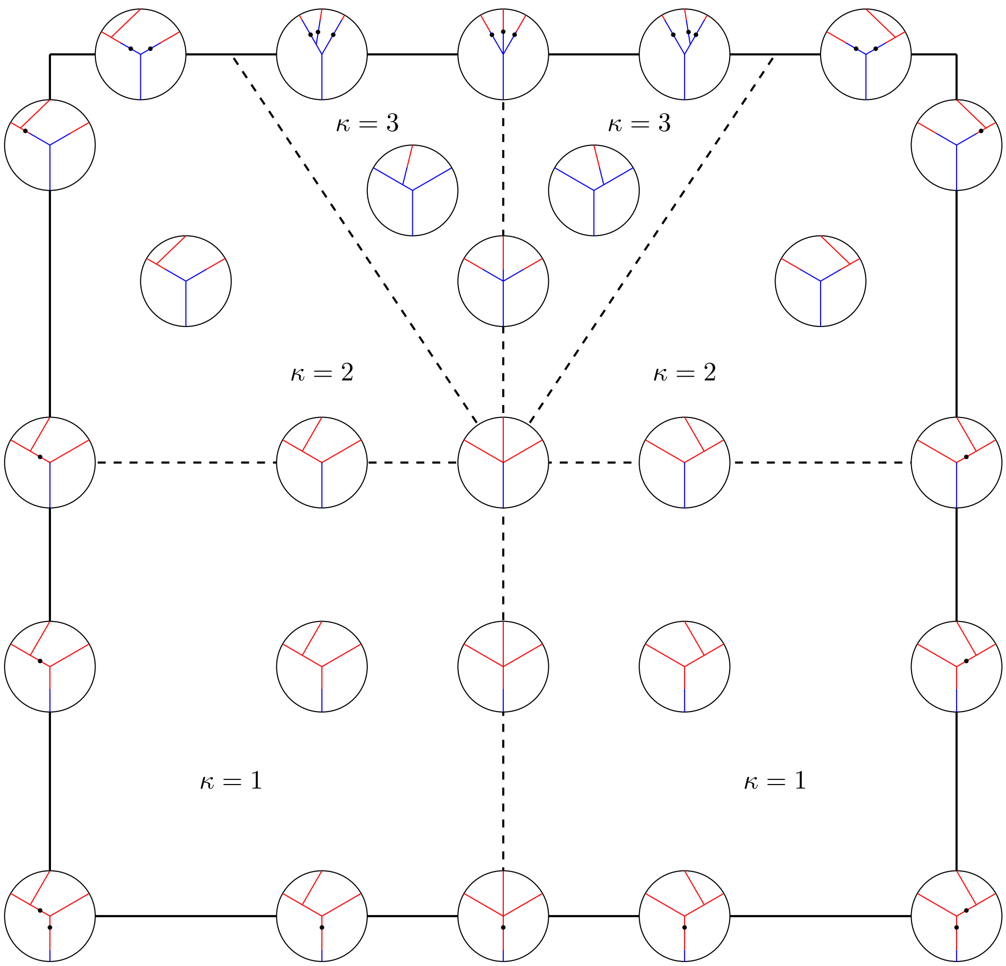
Definition 6. *Let (T, ℓ) be a metric tree. Let p be a point on the tree, not contained in the edge adjacent to the root. The* length of a point *$p \in T$ is the (possibly negative) length of the path from p to the vertex whose neighborhood contains the root, which we will denote $\ell(p)$. A* bicolored tree *is a tree metric tree with a selection of parameter (T, ℓ, τ). If T is stable, and if there is no vertex $v \in T$ with $\ell(v) = \tau$, then we call (T, ℓ, τ) a* stable bicolored tree*. To a stable bicolored flow tree we can associate the following data:*

- *A selection of points $p \in T$ so that $\ell(p) = \tau$. We will call these the* transition points *of the bicolored tree. The number of transition points will be denoted $\kappa(T, \ell, \tau)$.*
- *A partitioning of the leaves of T given by the connected components of $T \setminus \{v \in V \mid \ell(v) < \tau.\}$. The indexing for the partition will be*
$$(I_1|I_2|\cdots|I_\kappa) = \{1, \ldots, \nu(T)\}.$$
There should be one transition point for each partition; we will order the transition points p_i by their partition.

We call this selection of points a *bicoloring* as it separates the tree into two halves: those of points which have energy greater than the p_i, and those of points which have less length. Notice that the moduli space of j-bicolored trees is $(a, b) \times \mathcal{M}_T$, as the parameter of bicoloring length determines where the coloring switches. We will skip the steps to prove that this assembles into a nice space, as the proof should be completely analogous to the construction of $\mathcal{M}_k$.

Theorem 7. *The space of bicolored ribbon metric trees of valency k is a smooth manifold $\mathcal{B}_k$.*

The cell decomposition of this manifold is a little bit unclear; the combinatorial type of T does not determine the cell. Such a cell is additionally subdivided based on inequalities involving the time τ and the lengths γ which determine the number of transition points. For ease of notation, we will write $\mathcal{B}_{(T,\ell,\tau)}$ to denote the cell which contains a bicolored tree of this fixed type.
There are now two kinds of non-compactness that occur; those that come from one of the lengths of the edges going off to infinity, and a new boundary phenomenon that occurs from the length of the bicoloring going off to either $+\infty$ or $-\infty$. As a result, we now have many new boundary configurations. When length from bicoloring goes off to infinity, it is possible that many edge simultaneously break. For example, the configurations at the top of this diagram all come from the energy bubbling off to $+\infty$, and have 2 or 3 breaking points. The configurations on the left and right side come from the internal edge going breaking, and the configurations on the bottom of this diagram correspond to edges breaking as the bicoloring energy goes to $-\infty$. picture



A key observation is that whenever a blue edge goes to infinite length, this also corresponds to the length of the bicoloring going off to infinity. The moduli space of bicolored trees of valency k can be assembled by gluing together j-bicolored trees of valency k together by their energies. When constructing the compactification of the moduli space of bicolored metric trees by including broken bicolored metric trees, where the important factor is whether the length of the bicolored points goes off to infinity or not; roughly, we get two kinds of boundary components, corresponding to breaking that occurs before the color change, and breaking that occurs after the color change.

Theorem 8. *There is a compactification of $\bar{\mathcal{B}}_{||}$ into a manifold with corners. In this compactification the one dimensional strata can be described by*

$$\left(\bigcup_{1\leq i\leq j\leq k}\mathcal{B}_{j-i+1}\times\mathcal{M}_{k-j+i}\right)\cup\left(\bigcup_{1=i_1<\dots i_l=k}\mathcal{M}_l\times(\mathcal{B}_{i_2-i_1-1}\times\cdots\mathcal{B}_{i_l-i_1-1})\right)$$

Let's describe these two strata: one should think of the left term as corresponding to bicolor trees that broke in the blue segment, and therefore the length of the bicoloring does not go off to infinity. The more complicated term on the right corresponds to when the length of the bicoloring goes to infinity; as a result, possibly many red trees can break off. We'll want to take the combinatorics of bicolored trees and encode it in the geometry of a Morse function.

2.3 Flow Bicolored Trees

Bicolored trees come with more data than trees; the presence of the juncture gives us internal evaluation maps from the bicolored tree to the target space of the map.

Definition 7. *Let (T,ℓ,τ) be a bicolored tree, and let $j=\kappa(T,\ell,\tau)$ denote the number of transition points, which we will label $p_1,\dots,p_j\in T$. Let $H:X\to\mathbb{R}$ be a Morse function, and $V_1,\dots,V_j\subset X$ be submanifolds. Let $c_0,\dots,c_k$ be critical points of f. A (T,ℓ,τ) flow tree bicolored at $V_1,\dots,V_j$ is a (T,ℓ) flow tree in $\mathcal{M}_k(c_0;c_1,\dots,c_k)$ with the additional restriction that each bicolor point is mapped to a submanifold*

$$u(p_i)\in V_i.$$

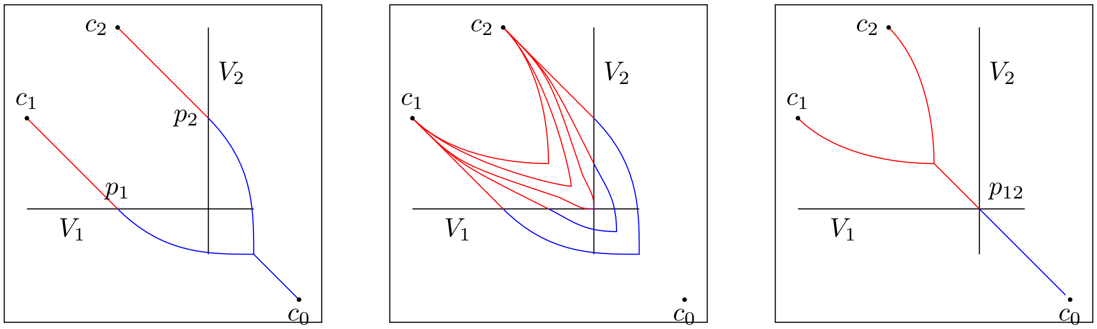
While we cannot guarantee any good breaking behavior if a critical point of H lies on V, when the critical points of X are disjoint from the submanifold V, in which case many of the facts we know about flow trees carry over.

Claim 6. *Fix (T,ℓ,τ) as above, as well as $V_1,\dots,V_\kappa$ various transition submanifolds. Suppose that* $\mathrm{Crit}(H)\cap V_i$ *is empty. Let $\mathcal{B}_{(T,\ell,\tau),V_i}(c_0;c_1,\dots c_k)$ be the set of $(T',\ell',\tau')\in B_{(T,\ell,\tau)}$ flow trees. Then $\mathcal{B}_{(T,\ell,\tau),V_i}(c_0;c_1,\dots,c_k)$ is a smooth manifold of dimension*

$$\left((2-k)-\operatorname{ind}c_0+\sum_{i=1}^{k}\operatorname{ind}(c_i)\right)+\left(1-\sum_{i=1}^{j}\mathit{codim}(V_k)\right).$$

There are a few interesting to note in this theorem. The first is the dimension of the moduli space. The first portion is simply the dimension of the moduli space of flow trees; the subtraction of codimension is the forced incidence condition of points. The extra parameter of τ increases the dimension by 1.
There are several boundaries for boundaries of $\mathcal{B}_{(T,\ell,\tau),V_i}(c_0;c_1,\dots,c_k)$. A large component of the boundary comes from the cell $\mathcal{B}_{(T,\ell\tau)}$ where τ approaches ℓ in such a way that the number of transition points $\kappa(T,\ell,\kappa)$ changes. When this happens, two or more of the of the transition points p_i,p_{i+1} will approach eachother, and so the resulting unstable bicolored flow tree will have a transition point mapping to $V_i\cap V_{i+1}$. This will become a notational nightmare for us in a moment, but until then, here is a picture motivating what

happens at these transition times.



Let's first introduce a piece of notation.

Notation 1. *Let $\mathcal{V} = \{V_1, \ldots V_k\}$ be a set of transversely intersecting manifolds. Let $I \subset \{1, \ldots, k\}$. Define $V_I = \bigcap_{i \in I} V_i$, and whenever $J = \{i, \ldots, j\}$, define the* contracted set of submanifolds

$$\mathcal{V}_{\cap I} := \{V_1, \ldots, V_{i-1}, V_J, V_{i+1}, \ldots, V_k\}$$

and the restricted set of submanifolds

$$\mathcal{V}_{|I} := \{V_i\}_{i \in I}$$

The diagram suggests we should define the following objects.

Definition 8. *Pick k, a valence, and select a selection of transversely intersecting submanifolds*

$$\mathcal{V} = \{V_1, \ldots V_k.\}$$

Recall that given a flow tree (T, ℓ, τ), we can select a subset of leaves $I_j \subset \{1, \ldots, k\}$ for each transition point p_j. A bicolored flow tree with valency k and transition points inherited from the $\mathcal{V}$ is a bicolored flow tree (T, ℓ, τ) where the point p_j is labeled by the submanifold V_{I_j}.

Theorem 9. *The space of k-valent bicolored flow trees with transitions inherited from $\mathcal{V}$ is a smooth manifold of dimension*

$$\left((2-k) - \operatorname{ind} c_0 + \sum_{i=1}^{k} \operatorname{ind}(c_i)\right) + \left(1 - \sum_{i=1}^{k} codim(V_i)\right)$$

We'll denote this moduli space $\mathcal{B}_{\mathcal{V}}(c_0; c_K)$.

It might be unexpected that we have consistency of dimension here, but as we lower the number of constraints coming from evaluation maps, we increase the codimension of the constraining submanifold, giving us a constant dimension across the moduli space. After this argument, we expect the following boundary behavior:

Theorem 10. *The moduli space $\mathcal{B}_{\mathcal{V}}(c_0; C_K)$ admits a compactification by broken bicolored trees:*

$$\partial \bar{\mathcal{B}_{\mathcal{V}}}(c_0; C_K) = \bigcup_{(I_1|J|I_2)=K} (\mathcal{B}_{\mathcal{V}_{\cap J}}(c_0; c_{I_1}, d, c_{I_2}) \times \mathcal{M}(d; c_J))$$
$$\cup \bigcup_{(I_1|\cdots|I_l)=K} \begin{pmatrix} \mathcal{M}(d_1, \ldots, d_l) \\ \times \\ \mathcal{B}_{\mathcal{V}_{|I_1}}(d_1, C_{I_1}) \times \cdots \times \mathcal{B}_{\mathcal{V}_{|I_l}}(d_l, c_{I_l}) \end{pmatrix}$$

While the boundary decomposition looks suitably terrifying, this exactly the boundary combinatorics that we had for the space of bicolored trees, where I've separated the two kinds of breaking (bicoloring length vs. edge length) going to infinity in two separate terms, combine with the necessary data of remembering how the constraints $\{V_i\}$ are inherited to the broken configurations.
We can assemble counts of bicolored flow trees to assemble some algebraic maps.

Definition 9. *Let (X, H) be a manifold with Morse function, and let $\mathcal{V}$ be a selection of transition submanifolds disjoint from the critical points. Define the* bicolor maps transitioning at $\mathcal{V}$ to be

$$b^k_{\{V_i\}} : C^{\otimes k}(X, H) \to C(X, H)$$

by the structure coefficients

$$\langle b^k_{\mathcal{V}}(c_1, \ldots, c_k), c_0 \rangle := \#\mathcal{B}_{\mathcal{V}}(c_0; c_K).$$

Non-surprisingly, this satisfies an algebraic relation coming from the combinatorics of the boundary strata of $\mathcal{B}$.

Theorem 11. *For each k, the bicolor maps satisfy the following algebraic relations:*

$$\sum_{(I_1|J|I_2)=K} b^{|I_1 \cup I_2|}_{\mathcal{V}_{\cap J}}(c_{I_1}, m^{|J|}(c_J), c_{I_2}) = \sum_{(I_1|\cdots I_l)=K} m^l \left(b^{|I_1|}_{\mathcal{V}_{|I_1}}(c_{I_1}), \cdots b^{|I_l|}_{\mathcal{V}_{|I_l}}(c_{I_l}) \right)$$

This is an ugly mess of algebra! One should be particularly bothered that each of the operations b^k is dependent on a bunch of choices of hypersurfaces. Let's move on to an application before we are forced to take on any more notation.

2.4 Application: Geometric Continuation maps

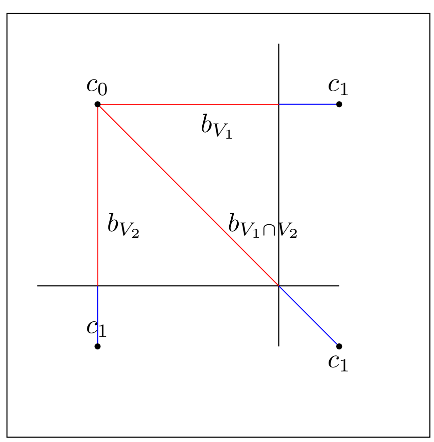
Let's look at the second of the relations given by flow bicolored trees,

$$b^1_{V_1 \cap V_2}(m^2(c_1, c_2) + b^2_{\{V_1,V_2\}}(m^1(c_1), c_2) + b^2_{\{V_1,V_2\}}(c_1, m^1(c_2)) = m^(b^2_{\{V_1,V_2\}}(c_1, c_2)) + m^2(b^1_{V_1}(c_1), b^1_{V_2}(c_2)$$

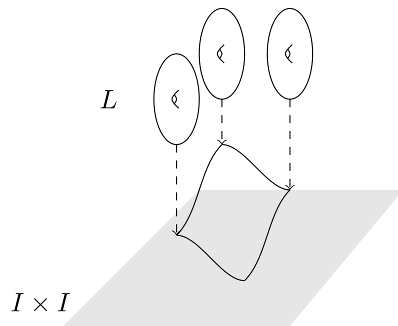
This would almost pass for an A_∞ homomorphism, if we didn't have all of these dependencies on V_i.
In order to get this to match up with our f^1 definition of the continuation map, we'll want to find a relation between the following terms

$$b^1_{V_1 \cap V_2} \sim b^1_{V_1} \sim b^1_{V_2} \sim f^1$$

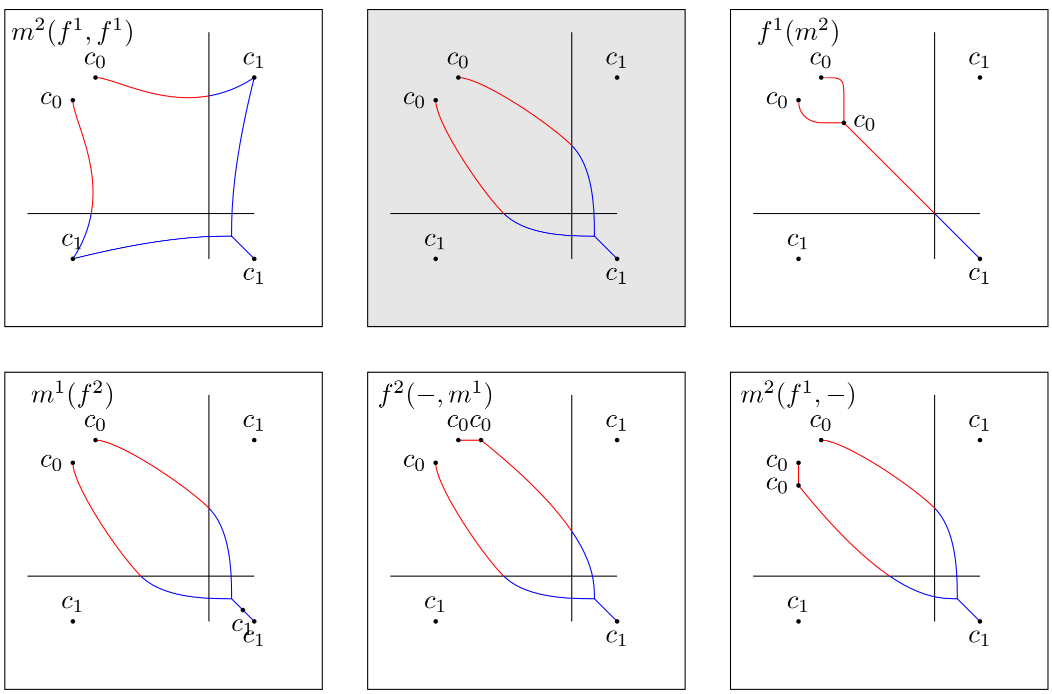
It is unreasonable for these three terms to be the same thing. Let's draw out a preliminary picture



While these three flows do not correspond to the same flow line, one could imagine that we equip $(L \times I \times I)$ with a suitable Morse function so that all three of these contribute to a count of flow lines giving us the A_∞ homomorphism relations.



Here is the inspiring picture, where the configurations in the gray box will contribute to the f^2 continuation map. The relations giving the 5 terms in the $k = 2$ relations can be realized as broken bicolored flow trees.



Basically the rest of this section is developing the notation and definitions to make this picture a proof of the A_∞ relations. Of particular interest to us should be the configuration corresponding to $m^2(f^1, f^1)$, where we should understand this as a product in the fiber spread out over a product in the base.

Definition 10 (This needs work)**.** *Let $H_0, H_1 : L \to \mathbb{R}$ be two Morse functions. A* consistent k-interpolation *between H_0 and H_1 is a choice of consistent $k-1$ interpolation datum, and a Morse function $H_{t_1,\dots,t_k} : L \times I^k \to \mathbb{R}$ satisfying the following conditions;*

- *(Non Critical) H only has critical points when $(t_1, \dots, t_k) \in \{0,1\}^k$.*
- *(Continuation) There exists ϵ so that whenever $t_i \geq 1 - \epsilon$, $H_{t_1,\dots,t_k} = H_1$*
- *(Consistency) There exists ϵ so that for each i, whenever $t_i \leq \epsilon$, we have $H_{t_1,\dots,t_k} = \hat{H}_{t_1,\dots,\hat{t}_i,\dots t_k}$, where $\hat{H}_{s_1,\dots,s_{k-1}}$ is the interpolation datum from the $k-1$ function.*

- *(Diagonal Consistency) In a small neighborhood of the diagonal, $H_{t,\dots,t} = H_t$.*

If $H_{t_1,\dots,t_k}$ is a consistent $k-1$ interpolation data, the critical points of $(L \times I^k, H_{t_1,\dots,t_k})$ correspond to copies of the critical points of H_0 and H_1. The critical points of $H_{t_1,\dots,t_k}$ in the base project to vertices of the cube, and we will label them x_I, where $I \subset \{1, \dots k\}$.

Lemma 1. *The critical points of $H_{t_1,\dots t_k}$ can be labeled as*

- *$c \in \operatorname{Crit} H_0$,*
- *$c \otimes x_I$, where the $I \neq$ and $c \in \operatorname{Crit} H_1$.*

The critical points belonging to $(\{0,1\} \setminus (0,\dots,0)) \times \operatorname{Crit}(H_1)$ have a very clean algebraic structure coming from the Kunneth formula, where I^k is given critical points so that its algebra is the alternating algebra generated by $x_1, \dots, x_k$.

Lemma 2. *Let $c_1, \dots, c_j \in \operatorname{Crit}(H_1)$. Then*

$$m^j_{H^k}(c_1 \otimes x_{I_1} \cdots , c_j \otimes x_{I_j}) = m^j_{H_1}(c_1, \dots, c_j)(t_J)$$

where

$$t_J = I_1 \sqcup \cdots I_j$$

if the sets I_l are disjoint.

Our configuration comes with a nice set of surfaces V_i given by

$$V_i := \{t_i = (1-\epsilon)\}.$$

Finally, we are in a good position to define the A_∞ homeomorphisms.

Definition 11. *Fix a consistent family interpolating data. Let $J \subset \{1, \dots, k\}$. Define the geometric continuation maps*

$$f^J : \operatorname{Crit}(H_0) \to \operatorname{Crit}(H_1)$$

by taking the composition

$$b^J_{\mathcal{V}_{|J}} : \operatorname{Crit}(H_0) \to \operatorname{Crit}(H^k) \xrightarrow{\pi_{x_J}} (H_0)$$

Also define

$$f^{I_1 \cup I_2} : \operatorname{Crit}(H_1) \to \operatorname{Crit}(H_1)$$

by taking the composition

$$b^{I_1 \cup I_1}_{\mathcal{V}_{\cap J}}.$$

Our consistency condition tells us that this definition only depends on $|J|$.

Claim 7. *If $|J| = |J'|$ then $f^J = f^{J'}$. Additionally, if $|I_1 \cup I_2| = J$, we have the same deal.*

Due to the independence from J, we will define $f^j : C(L, H_0)^{\otimes} j \to C(L, H_1)$ by selecting any such J.

Theorem 12. *The f^j satisfy the A_∞ homomorphism relations.*

Proof. The proof at this point is computation. To check the A_∞ homomorphism relation, we start with the relation on the set of bicolor maps

$$\sum_{(I_1|J|I_2)=K} b^{|I_1 \cup I_2|}_{\mathcal{V}_{\cap J}}(c_{I_1}, m^{|J|}(c_J), c_{I_2}) = \sum_{(I_1|\cdots I_l)=K} m^l \left(b^{|I_1|}_{\mathcal{V}_{|I_1}}(c_{I_1}), \cdots b^{|I_l|}_{\mathcal{V}_{|I_l}}(c_{I_l}) \right)$$

We now can make some simple substitutions.

- Whenever $m^{j-i}(c_J) \notin \mathrm{Crit}(H_0)$, the term

$$b^{|I_1 \cup I_2|}_{\mathcal{V}_{\cap J}}(c_{I_1}, m^{|J|}(c_J), c_{I_2}) = 0$$

  This is because the maps $b^{|I_1 \cup I_2|}_{\mathcal{V}_{\cap J}}$ are only nonzero when the inputs come from $\mathrm{Crit}(H_0)$. We may simplify the left hand side to

$$b^{|I_1 \cup I_2|}_{\mathcal{V}_{\cap J}} = f^{|I_1 \cup I_2|}(c_{I_1}, m^{|J|}_{H_0}(c_J), c_{I_2}).$$

- The right hand side is a little trickier. The bicolor map $b^{|I_1|}_{\mathcal{V}_{|I_1}}$ ends up computing some element of the form

$$(f^{I_1}(c_1, \ldots, c_{i-1}) + \text{Terms with } x) \otimes x_{I_1}$$

  By Lemma **??**, the higher x_i terms will disappear in the product. We may safely substitute $b^{I_1}_{\mathcal{V}_{|I_1}} \to f^{|I_1|} \otimes x_{I_1}$ in the right hand side. After making these substitutions,

$$\begin{aligned}\sum_{(I_1|\cdots I_l)=K} m^l \left(b^{|I_1|}_{\mathcal{V}_{|I_1}}(c_{I_1}), \cdots b^{|I_l|}_{\mathcal{V}_{|I_l}}(c_{I_l})\right) &= \sum_{(I_1|\cdots I_l)=K} m^l \left(f^{|I_1|}_{\mathcal{V}_{|I_1}}(c_{I_1}) \otimes x_{I_1}, \cdots b^{|I_l|}_{\mathcal{V}_{|I_l}}(c_{I_l}) \otimes x_{I_l}\right) \\ &= \sum_{(I_1|\cdots I_l)=K} m^l_{H^1} \left(f^{|I_1|}(c_{I_1}), \cdots, f^{|I_l|} c_{I_l})\right)\end{aligned}$$

This proves the A_∞ homomorphism relation. □

References

[Abo09] Mohammed Abouzaid. "A topological model for the Fukaya categories of plumbings". In: *arXiv preprint arXiv:0904.1474* (2009).

[AD13] Michéle Audin and Mihai Damian. *Morse Theory and Floer Homology.* Springer, 2013.

[Fuk96] Kenji Fukaya. "Morse homotopy and its quantization". In: *Geometric topology (Athens, GA, 1993)* 2 (1996), pp. 409–440.

[Hut02] Michael Hutchings. *Lecture notes on Morse homology (with an eye towards Floer theory and pseudoholomorphic curves).* Lecture Notes. UC Berkeley, 2002.

[Mes16] Stephan Mescher. "Perturbed gradient flow trees and A_∞-algebra structures on Morse cochain complexes". In: *arXiv preprint arXiv:1611.07916* (2016).