Contents

1 Getting Started: Denis Auroux

This seminar is on the symplectic geometry version of mirror symmetry, and we'll be focusing on Homological mirror symmetry. This means our viewpoint will be based around Kontsevich's Homological Mirror Symmetry conjecture.

1.1 Where does this conjecture come from?

In the 1990's, the constraints of symmetry led string theorists to conclude that the universe was modeled by a 10 dimensional manifold. This consists of a 4 dimensional flat component (which gives the large space-time dimension that we see,) and a Calabi-Yau 3-fold X. At the expense of reducing the predictive power of the theory, they produced 2 interesting mathematical models of string theory – which they call the A and B models of string theory. Each of these models study different kinds of submanifolds of X called branes but roughly:

- The A model studies the symplectic geometry of X.
- The B model studies the algebraic geometry of X.

If we fix a Calabi-Yau X, the A and B models are quite different. However, it is believed that for every X there is a new manifold, $X^\vee$, for which the A model on X corresponds to the B model on $X^\vee$.

Conjecture 1 (Mirror Symmetry)**.** *For every Calabi-Yau X, there is a mirror manifold $X^\vee$ such that symplectic invariants on X correspond to algebraic geometry of $X^\vee$, and vice versa.*

While Mirror Symmetry now studies many different families of manifolds, it started with Calabi-Yau 3-folds.

Definition 1 (Calabi-Yau)**.** *A* Kähler manifold *is a complex manifold (X, J), with a 2-form ω which satisfies the following conditions:*

- $d\omega = 0$
- *$\omega(v, Jw)$ is a metric on the space.*

If we can pick a holomorphically non-vanishing $\Omega \in \Omega^{n,0}(X)$, then we say that X is (weakly) Calabi-Yau. If in addition $|\Omega|_g$ is constant, we say that X is strongly Calabi-Yau.

As a generalization, let X be a suitable Kähler manifold. Then the mirror $X^\vee$ would be a Kähler manifold with $W : X^\vee \to \mathbb{C}$ a global holomorphic function. Note that if W is nonconstant, then $X^\vee$ is not compact. The name for this generalization is called a *Landau-Ginzburg Model.*

Historically, it was thought that there were only a few examples of Calabi-Yaus. One of the first known examples was that of the quintic threefold in $\mathbb{CP}^4$.

Example 1 (Complex Tori)**.** *Let $\Lambda \subset \mathbb{C}^n$ be a lattice. Then $\mathbb{C}^n/\Lambda$ is a Calabi-Yau* [1]

Example 2 (Quintic Threefold)**.** *The locus $\sum_{i=0}^4 z_i^5 = 0$ in $\mathbb{CP}^4$, is a Calabi-Yau 3-fold.*

In physics, each Calabi-Yau gives a different model for physics and this led to a search for new Calabi-Yaus. It was initially hoped that there would be a small number of Calabi-Yau 3-folds, but many were eventually found. It was noticed that when a Calabi-Yau manifold X was found, there was usually another Calabi-Yau $X^\vee$ with interchanged Hodge diamond data. The name "Mirror Symmetry" comes from this interchange in the Hodge diamond. For a Calabi-Yau 3-fold, $H^{1,1}(X)$ is a moduli space of Kähler forms, and $H^{1,2}$ is a moduli space of complex structures. The mirror correspondence is suppose to switch these

[1] Some folks add the extra condition that X be simply connected to the definition of Calabi-Yau.

moduli spaces.
By 2000, [KS00] had found 30,108 different pairs of Hodge number data, and the construction used produced many of these mirror pairs.
Closed-string mirror symmetry was predicted a matching of enumerative invariants between symplectic and algebraic geometry. The A model was interested in counting the number of almost complex curves representing a class in homology, while the B model looked at periods of holomorphic functions coming from Hodge theory. In 1990, Candelas, Ossa, Green and Parks predicted that these invariants were interchanged by mirror symmetry. While periods of holomorphic functions could be computed, counting holomorphic curves was a difficult problem. In [CXGP91] , they produced accurate counts of the number of degree k curves in the quintic threefold. Their method relied on physical intuition that took some time to make mathematically rigorous. The framework for these theories were fleshed out by Givental [Giv99] , and Lian-Liu-Yau [LLY97]. This flavor of mirror symmetry is called "closed string mirror symmetry." Here are some geometric attributes conjectured to be switched by the mirror correspondence:

A model on X	B model on $X^\vee$
Symplectic Structures and deformations	Complex Structures and Deformations
$h^{1,1}(X)$	$h^{1,2}(X)$
Gromov-Witten Invariants	Gauss-Manin Connection
Fukaya Category	Derived Category of Coherent Sheaves

Kontsevich in [Kon94] gave a precise meaning to what the open-string version of mirror symmetry. In our case, we will be looking at the case of disks.

Conjecture 2 (Kontesvich HMS, 1994-)**.** *Suppose that X and $X^\vee$ are a mirror pair. Then $\mathcal{F}(X,\omega)$, the Fukaya category, of X and $\mathrm{Coh}(X^\vee)$, category of coherent sheaves, are derived equivalent.*

What are these different categories? Roughly:

- On the A-side, the Fukaya Category of X has as objects Lagrangian submanifolds whose morphisms are given by taking counts of intersections between them. To make this rigorous, we will have to take a count up to Lagrangian Floer Homology,

$$\hom(L_0, L_1) = CF^*(L_0, L_1).$$

  Composition of morphisms will be done by counting disks with boundaries on 3 Lagrangians. This will not be associative, but will satisfy the A^∞ relations.

- On the B side, we look at coherent sheaves. During this seminar, we won't go into much depth about coherent sheaves, but here is one intuition that can be used. Coherent sheaves are the sheaves which can be constructed as cokernels of maps between holomorphic vector bundles. If you think of these as submanifolds, you want to look at not just the intersections of these, but also the homology of these with coefficient in the holomorphic vector bundles.

Why do we look at the derived versions of these categories? The basic reason is that we don't know how to define things like kernels and cokernels are in the Fukaya category, and the derived category enlarges the Fukaya category with mapping cones giving it kernels and cokernels. On the Coherent sheaf side, this allows us to only thing about holomorphic vector bundles.

1.2 A timeline for Homological Mirror Symmetry

- 1998: Polishchuk-Zaslow prove mirror symmetry for T^2. [PZ98]
  The symplectic geometry of T^2 is quite simple: the only important quantity is $\int_{T^2}\omega$, which is given by a single number. On the mirror side, we have the torus now viewed as an elliptic curve, $E = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. In this case, the mirror pairing is given by $\tau = iA$. The correspondence is given by

- L_0 corresponds with $\mathcal{O}$, the structure sheaf.
- L_p corresponds to the skyscraper sheaf $\mathcal{O}_p$.
- $L_{a,b}$ corresponds to a degree 1 line bundle , where a and b describe the homology class of this Lagrangian submanifold.

In T^2 the correspondence can be checked by hand. For instance:

- $\hom(L_0, L_1)$ is $\mathbb{C}$, because there is a one dimensional space of morphism, as these Lagrangians intersect at a single point. This matches on the nose with $\hom(\mathcal{O} \to \mathcal{L}) = \mathbb{C}$.
- $\hom(L_1, L_p)$ is similarly $\mathbb{C}$, because there is a one dimensional space generated by the intersection point, and it again matches with $\hom(\mathcal{L} \to \mathcal{O}_p)$.
- The composition map on the B model side corresponds to the theta function $\theta(\tau, z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z}$, which is almost τ periodic. On the A model side, there are many triangles that have boundaries linking the 3 intersection points. If counted with weights $e^{-\text{area}}$, these triangles calculate out the value of the θ function.

This is the first good piece of evidence for homological mirror symmetry.

- 2001: Kontsevich - Soibelman start looking at T^4 through torus fibrations [KS01] Abouzaid and Smith finished the program in 2008 [AS+10] for the four-torus.
- 2003: Seidel proves Homological mirror symmetry for $K3$ [Sei03b] using the machinery Picard Lefschetz theory, [Sei08].
- 2011: Sheridan proves it for the quintic 3-fold [She11].

Outside of the Calabi-Yau case, there has been a surprising amount of success. Versions of homological mirrors symmetry have been proven for $\mathbb{CP}^1, \mathbb{CP}^2$, blowups and toric varieties.

1.3 Common Techniques

Initial attempts look at actually computing out the Fukaya category of a manifold, and matching the categories on the nose. In general, we have no idea what the Lagrangian submanifolds of a manifold are. For example, we do not even know what the Lagrangians are inside $\mathbb{R}^6$. By Darboux's theorem, we know that we can find a copy of symplectic $\mathbb{R}^6$ inside every Calabi-Yau 3-fold.
However, there is hope for calculating Fukaya categories. Homological mirror symmetry states that we are not interested in Lagrangians of a manifold; we are only interested in their Floer homology. We can use this to our advantage by using the tools of homological algebra to extract large amounts of data about $\mathrm{Fuk}(X, \omega)$ based on some choice Lagrangians.
When we look at the derived category of the Fukaya category, we can look at a small set of generating Lagrangians which give the whole derived category through iteration of the mapping cone process. For example, on the elliptic curve, L_0 and L_p can generate all of the other Lagrangians through this mapping cone process. The good news is that we can show that we can find a set of Lagrangians that generate, and completely understand the derived Fukaya category. The general scheme to prove that these Lagrangians can generate is to find for every Lagrangian L a diagram of the form

$$\begin{tikzcd} L \arrow[rr, "\mathrm{id}"] \arrow[dr, "\exists"'] & & L \\ & \text{Generators } G_i \arrow[ur, "\exists"'] & \end{tikzcd}$$

This also gives us a way to construct the mirror symmetry correspondence. Suppose we can find some generating sets of $\mathcal{F}$ and $\mathcal{D}^b\operatorname{Coh}(C^\vee)$, and check that the A^∞ morphisms match up. This means that we need to check a lot of morphisms, which is hard! The way we get around this is by using deformation theory to classify $A-\infty$ algebras. This can be computed using Hochschild cohomology.

2 Lagrangian Floer Cohomology: Andrew Hanlon

The main source for this talk is [Aur14]. In this talk, we will use the following notation:

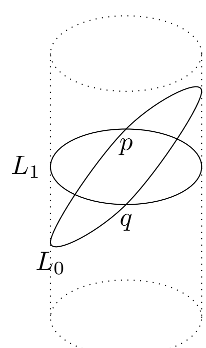
- (M, ω) is a symplectic manifold.
- Recall, $L \subset M$ is called *Lagrangian* if $\omega|_{L^n} = 0$, and $\dim L = \frac{1}{2} \dim M$. Some examples include $\mathbb{R}^n \subset \mathbb{C}^n$ with the standard symplectic form, or the image of a one form in the cotangent bundle with standard symplectic structure. We will today assume that Lagrangians are compact.
- If $H \in C^\infty(M \times [0,1], \mathbb{R})$ is a function, we define it's *Hamiltonian vector field* X_H with the convention
$$\omega(\cdot, X_{H_t}) = dH_t,$$
and $\psi = \psi_1$ to be the time one flow of the Hamiltonian vector field.
- We also have a choice of almost complex structure $J : TM \to TM$ which is compatible with the symplectic structure. We define the metric
$$g(v, w) := \omega(v, Jw).$$
The space of such J is contractible and non-empty.

The motivation for Floer theory comes from Arnold's conjecture on the number of fixed points of ψ.

Theorem 1 (Arnold Conjecture, [Flo88])**.** *If the symplectic area of any disc in M with boundary in L vanishes and $L \pitchfork \psi(L)$, then*
$$|L \cap \psi(L)| \geq \sum_i H^i(L, \mathbb{Z}/2\mathbb{Z}).$$

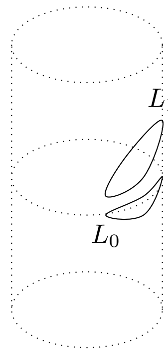
The example that we will constantly return to is Lagrangians in the symplectic cylinder $M = T^*S$.

Example 3 (Symplectic Cylinder)**.** *Let M be the cylinder and let L be a meridian of the cylinder. Then every hamiltonian flow of L will have 2 intersection points. This follows from the for this particular choice of L, the signed area between L and $\psi(L)$ is zero.*



In particular, you cannot displace the equatorial Lagrangian from itself. Notice that L satisfies the condition as L bounds no disks in M.
However, if L does bound a disk, we have a counterexample where an Hamiltonian isotopy of L and itself have no common intersection. For the same isotopy in the previous example, we have the following pair of

non-intersecting Lagrangians.



Our sketch of the proof of 1 will involve a homology theory whose differential counts pseudoholomorphic disks. Here is an outline of the rest of these notes:

- In 2.1, we describe the moduli space of pseudoholomorphic strips. In suitably nice cases, we will show that this space is compact up to a phenomenon called bubbling.
- In 2.2 we set up the Floer cochain complex. Let L_0, L_1 be two Lagrangians that intersect transversely. Define $CF(L_, L_1)$ to be the free $\mathbb{Z}_2$ module generated by the intersection points of L_0 and L_1. This is equipped with differential $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$ which counts the number of J holomorphic strips that flow from one intersection point to the next.
- In 2.3, we describe how to index the Floer cochain complex based on the geometry of the Lagrangians.
- 2.4 proves that if L_1 and L_1' are Hamiltonian isotopic, there exists chain homotopy between $HF(L_0, L_1$ and $HF(L_0, L_1')$. We also outline an isomorphism between $HF(L, L_1) = H^*(L)$ when L_1 and L are hamiltonian isotopic.
- 2.5 describes what can be done in the case when the Lagrangians bound disks.

2.1 Holomorphic Curves

Definition 2 (J-holomorphic Strips)**.** *Let L_0 and L_1 be two Lagrangian submanifolds of L. A J-*holomorphic strip *between L_0 and L_1 is a map*

$$u : \mathbb{R} \times [0,1] \to M$$

satisfying the following properties:

- Pseudoholomorphic $\bar{\partial}_J u = 0$.
- Boundary Conditions $u(s,0) \in L_0$ *and* $u(s,1) \in L_1)$.
- Bounded Energy $E(u) = \int_{\mathbb{R}\times[0,1]} u^*\omega < \infty$.

We say that this strip runs from p to q if $\lim_{s\to\infty} u(s,t) = p$ and $\lim_{s\to-\infty} u(s,t) = q$.[2]

The linearization $D_{\bar{\partial}_J, u}$ at a solution u is a Fredholm operator on some appropriate Banach space.

Definition 3 (Moduli Spaces)**.** *let $\hat{\mathcal{M}}(p, q; [u], J)$ be the space of J holomorphic strips from p to q in the same class of $[u] \in \pi_2(M, L_1 \cup L_1)$. There is an $\mathbb{R}$ action on this space by translation, and we quotient our moduli space by this reparameterization*

$$\mathcal{M}(p, q; [u], J) := \hat{\mathcal{M}}(p, q; [u], J)/\mathbb{R}.$$

[2]Note the weird direction here. This is because we are defining Floer Cohomology. In the homology, the strips run in the other direction.

Theorem 2. *If $D_{\bar{\partial}_J,u}$ is surjective, then $\hat{M}$ is a manifold, and* $\dim \hat{M} = \int D_{\bar{\partial}_J,u} := \operatorname{ind}[u]$.

There are 2 nice properties that we would like $\mathcal{M}$ would have: compactness and orientation. Compactness comes from this condition on the non-existence of disks with symplectic area, and orientation can be determined by putting spin structures on our Lagrangians. We will assume that we are working in the case where $\mathcal{M}$ has these properties, so that $\mathcal{M}(p,q);[u],J)$ is a compact 0-dimensional manifold if $\operatorname{ind}(u) = 1$. We want to count these strips taking into account their energy. To do so, we need to use a slightly strange ring for our coefficients.

Definition 4. *Let $\mathbb{K}$ be a field. Define the* Novikov ring *over k to be*

$$\Lambda_0 := \{\sum_{i=0}^{\infty} : a \in \mathbb{K}, \lambda_i \geq 0, \text{ and } \lim_{i\to\infty} \lambda_i = +\infty\}.$$

The Novikov field *is similarly defined $\Lambda = \operatorname{Frac}(\Lambda_0))$. In terms of power series, it is*

$$\Lambda_0 := \{\sum_{i=0}^{\infty} : a \in \mathbb{K}, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = +\infty\}.$$

2.2 Floer Theory

Definition 5. *Let M be a symplectic manifold satisfying the above conditions, and L_0, L_1 be two Lagrangian submanifolds. Define $CF(L_0, L_1)$ to be the set of intersection points of L_0 and L_1. Let $p \in L_0 \cap L_1$.*

$$\partial p := \sum_{\substack{q \in L_1 \cap L_1 \\ \operatorname{ind}[u]=1}} \#\mathcal{M}(p,q;[u],J))T^{\omega([u])}q$$

where $\omega([u]) = \int u^*\omega$, and the count $\#$ is being made with sign chosen by orientation.

The use of Novikov coefficients is so we can count infinitely many disks. Gromov compactness states that there are finitely many disks of bounded symplectic area, so if we have infinitely many disks, they must diverge in symplectic area and therefore give us a count in the Novikov field. In the exact case, where $\omega = d\alpha$, and $\alpha|_{L^i} = df_i$, we can work over our base field $\mathbb{K}$.

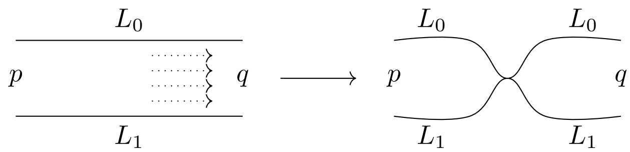
Theorem 3 ([Flo88])**.** *If $[\omega] \cdot \pi_2(M, L_i) = 0$, then ∂ is well defined with respect to some J, $\partial^2 = 0$, and $HF(L_0, L_1)$ is independent of Hamiltonian isotopies and choice of J.*

Sketch of Proof. We need to address 3 issues for the moduli space:

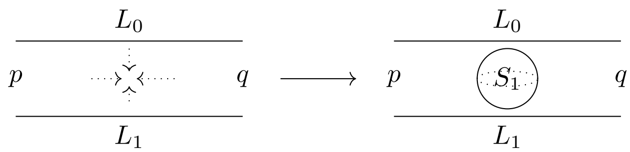
1. Transversality of $D_{\bar{\partial}}$.
   If $L_0 \pitchfork L_1$, then it's enough to take a generic time dependent J_t. If you are willing to work with a time dependent J, the generically you will have the surjectivity of $\frac{\partial u}{\partial s} + J(u,t)\frac{\partial u}{\partial y} = 0$.
   If L and L_1 are not transverse, we can also take a generic Hamiltonian $H \in C^\infty(M \times [0,1])$, and then perturb the equation to count solutions to $\frac{\partial u}{\partial s} + J(t,u)\left(\frac{\partial u}{\partial t} - X_H(u,t)\right) = 0$. Unfortunately, we are no longer generating $CF(L_0, L_1)$ by intersection points- we are counting flow lines that start on L_0 and ending on L_1. Such a flow line is the same as having an intersection between L_0 and $\psi^{-1}L_0$, so if we can show that this thing is independent of hamiltonian perturbations of Lagrangians, we will recover a count of intersection points.

2. Compactness of the Moduli Space.
   Gromov's compactness theorem tells us that a sequence of J-holomorphic strips converges to a tree of J-holomorphic disks and spheres. To see all the possibilities, need to look at where energy blows up on the domain. In our case, there are 3 possibilities:

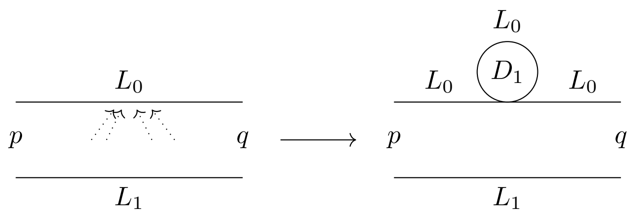
- If energy builds up at either end of the domain strip, we get strip breaking.



- If energy builds up at an interior point, we get a sphere bubble.



- If energy builds up at the boundary of the domain, and we get a disk bubble. This is very bad!



For us, we will ignore the second two kinds of bubbling behavior; our condition on the symplectic energy of disks with boundaries in our Lagrangians forbids the formation of these bubbles.

3. Orientation.
   We will ignore this issue, and just hope that we have the signs right.

Now assuming the above, let $[u]$ be a class with $\mathrm{ind}([u]) = 2$. Gromov compactness produces $\bar{\mathcal{M}}(p,q;[u],J)$ a compact one-dimensional manifold. Furthermore, the compactification has boundary strata which admits a nice description:

$$\partial\bar{\mathcal{M}}(p,q);[u],J) = \bigsqcup_{\substack{r\in L_0\cap L_1\\ [u']+[u'']=[u]}} \mathcal{M}(p,r;[u'],J)\times\mathcal{M}(r,q;[u''],J).$$

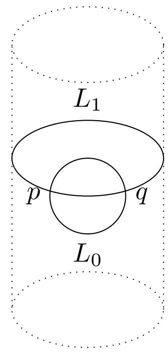
where you should think of this as being the two strips that u breaks into in the Gromov compactification. This behavior goes the other way as well. Suppose we have two more classes decomposing $[u]$ as $[u']+[u'']=[u]$. Since the index is additive, we haveind$[u'] = \mathrm{ind}[u''] = 1$. The Gluing lemma implies that all such strips u' and u'' can be realized as a limit of curves $u_i \in [u]$ which breaks into two strips in the Gromov compactification.
Therefore, the coefficient of q in $\partial^2 p$ is exactly

$$\sum_{\substack{r\in L_0\cap L_1\\ [u]:\mathrm{ind}[u]=2\\ [u']+[u'']=[u]}} (\#\mathcal{M}(p,r;[u'],J))(\#\mathcal{M}(r,q;[u''],J)T^{\omega[u']+\omega[u'']}.$$

Since this is the count with sign of the boundary of a collection of one dimensional manifolds, the count comes to 0. $\square$

The necessity of our conditions is best given by a counterexample:

Example 4 (Counterexample to $\partial^2 = 0.$). *Consider in the cylinder two Lagrangians; one given by the equator and another circle which intersects it at two points.*



There are all kinds of holomorphic disks here, and they do not come in the required cancelling pairs!

2.3 Index of Strips

Let $LGr(n)$ be the Lagrange Grassmanian of $\mathbb{R}^{2n}$ with standard symplectic form. There is diffeomorphism between $U(n)/O(n)$. It follows from long exact sequence on homotopy that $\pi_1(LGr(n)) \simeq \mathbb{Z}$, and in fact $\det^2 : U(n)/O(n) \to S^1$ induces the isomorphism.

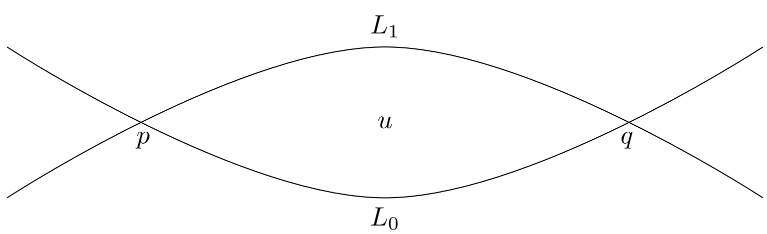
Definition 6 (Maslov Index). *Let $\ell(t)$ be a loop in $LGr(n)$. The* Maslov index *of ℓ is defined to be the winding number of* $\det^2$.

Let M be the ambient manifold, and $u : \mathbb{R} \times [0,1] \to M$ be a J holomorphic curve. Since $\mathbb{R} \times [0,1]$ is contractible, we can trivialize u^*TM over the domain of the strip. This gives us two paths in the Lagrange Grassmanian: $\ell_i u^*|_{\mathbb{R}\times\{i\}} TL_i$, where i is 0 or 1. We want to connect these two paths up, which will involve making some choices.
For any two Lagrangian subspaces L, L' in $\mathbb{C}^n$ such that $L \cap L' = 0$, then there is a unique matrix $A \in Sp(2n)$ such that $A(L) = (\mathbb{R}^n)$ and $A(L') = i\mathbb{R}^n$. The *canonical short path* from L to L' is $\lambda(t) = A^{-1}(e^{i\pi t/2}\mathbb{R}^n)$. Some intuition is that this is the minimal way to map two Lagrangians subspaces to each other in the clockwise direction.
Returning to our holomorphic strip from p to q, let λ_p be the canonical short path from T_pL_0 to T_pL_1, and the same for λ_q. Now consider the loop from T_qL_0 back to itself in the Lagrange Grassmanian:

$$-\ell_0 * \lambda_p \ell_1 * -\lambda_q.$$

Define the *Maslov index of u* to be the Maslov index of this loop.

Example 5. *Returning to our example of two Lagrangians in the plane.*



- *From p to q, we rotate in the positive direction.*
- *Taking the short path rotates back in the negative direction the same amount.*

- *From q to p rotates in the positive direction*
- *The short path rotates again in the positive direction*

In the end, we will end up going around once, so the Maslov index of this strip is one.

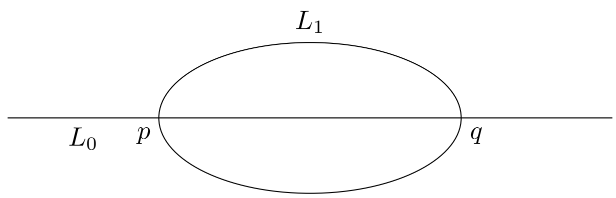
This can be used to give us a grading on the Floer Chain complex, provided we have the following two conditions:

- $2c_1(TM) = 0$, which will allow us to take a fiberwise universal cover $\pi : \widetilde{LGr}(TM) \to LGr(TM)$.
- For every Lagrangian we consider, we want the *Maslov Class* $\mu_L \in H^1(L, \mathbb{Z}) = \mathrm{Hom}(\pi_1(L), \mathbb{Z})$ to vanish. The Maslov class is the obstruction to having a lift of $L \to LGr(TM)$ to the universal cover $\widetilde{LGr}(TM)$. This is a pretty restrictive condition.

If both of these conditions hold, we can take graded lifts $\tilde{L}_0, \tilde{L}_1$ of the two Lagrangians. At $p \in L_0 \cap L_1$, take a path from $T_p\tilde{L}_0 \to T_p\tilde{L}_1$. This gives a distinguished homotopy class of paths $[\ell_p]$ from T_pL_0 to T_pL_1. Define the degree of p to be

$$\deg(p) := \mu(\ell_p * -\lambda_p).$$

Example 6 (Failure of Grading)**.** *In this counterexample, both the upper (from p to q) and lower (from q to p) strips have Maslov index one.*



Therefore, we cannot assign a grading to a Floer theory of these two Lagrangians. However, whenever L_0 and L_1 are oriented, we can create a $\mathbb{Z}_2$ grading based on orientation.

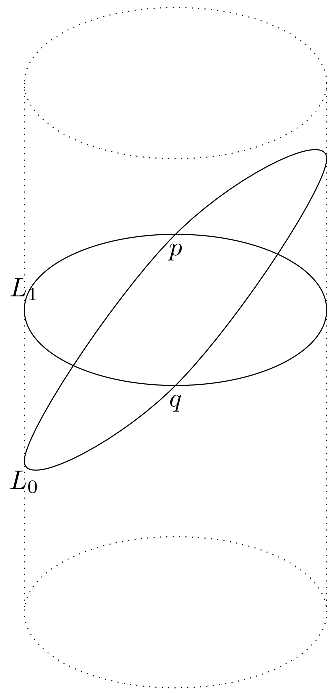
So, with these conditions, we have associated to a pair of Lagrangians L_1 and L_2 a chain complex with grading given by the Maslov index.

2.4 Recovering the cohomology of L.

Theorem 4. *Suppose that L is compact, $[\omega] \cdot \pi_2(M, L) = 0$, then*

$$HF(L, L) \simeq H^*(L, \Lambda).$$

Example 7. *Let's return to the example of T^*S^1 with the equator Lagrangian.*



There are two intersection points p and q. One can check that both of the strips are index 1 going from p to q, with opposite orientations. Therefore $\bar{\partial}p = \pm T^{\omega(u)}q + \mp T^{\omega[u]}q = 0$, so both points end up surviving in homology. This gives us $HF(L, L) \simeq H^*(S^1)$.

We can prove the theorem by generalizing the example. Let N be the Lagrangian given by the zero section of T^*N. Take f to be a Morse function, and ϵ a small number. Consider the function

$$\epsilon f \circ \pi : T^*N \times \mathbb{R}.$$

Hamiltonian flow is an isotopy from the zero section to a graph of ϵdf. Since f is Morse, the section ϵdf is transverse to the zero section, with intersection points given by the critical points of f. This gives us a correspondence between the generators of $CF^*(L, L)$ and $C^*(L)$.

We need a correspondence then between Morse and Floer trajectories. In Morse homology, we need some gradient trajectory

$$\dot{\gamma}(s) = \epsilon \nabla f(\gamma(s)),$$

and in Floer theory, we need

$$\frac{\partial u}{\partial s} + J(t, u)\frac{\partial u}{\partial t} = 0.$$

The idea of the correspondence is that $\gamma(s) = u(s, 0)$. Now, given small enough ϵ, or by careful enough choice of J, we can make this match exactly.

In the other direction, consider the Hamiltonian perturbed equation

$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_H\right) = 0$$

so trajectories are hamiltonian flows that start and end at the zero section. Now given a gradient trajectory $\gamma(s)$, we can set $u(s, t) = \gamma(s)$. Then after proving that the Floer theory is independent of Hamiltonian isotopy, we would get that Floer theory is isomorphic to the normal cohomology.

How do we deal with the case where N is not given by the zero section? Suppose that there was a holomorphic disk which goes well outside of a small neighborhood of N, no matter how small the hamiltonian perturbation. Then Gromov compactness will give you some kind of contradiction.

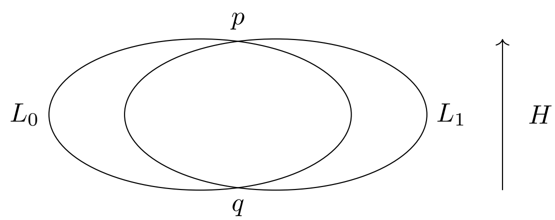
2.5 What about when you bound some Disks: Denis Auroux

The running example we want to work with is the circle in the plane.

Definition 7. *A Lagrangian L is called* monotone *if there is a constant K such that for all $[u] \in \pi_2(M, L)$,*

$$\int u^*\omega = K\mu(u).$$

The circle in the plane satisfies this property, as the only disks on this will be k-fold covers of it. Similarly, the equator on the sphere satisfies this property. Let's work through an example with the disk.

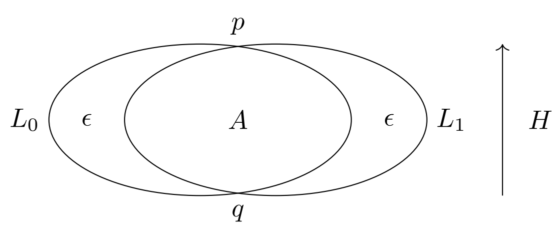


There are 3 holomorphic strips that we can easily see here: one on each of the two sides, and one in the middle which runs the opposite direction. Let ϵ be the area of the side parts, and A the area of the center strip. Then

$$\partial(p) = (T^\epsilon - T^\epsilon)q = 0$$

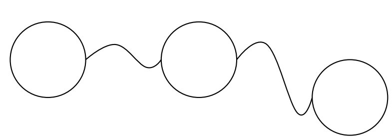
$$\partial(q) = \pm T^A p.$$

Here we have a problem. We know $HF(L, L) = 0$, because we can displace L from itself by Hamiltonian isotopy. However, we would like to recover the homology of the disk.
The idea is to break area between the Lagrangians into "thin" and "thick" areas. We will look at holomorphic strips which will have a small amount of area in the ϵ areas, and a large amount of area in the A in section.



When ϵ goes to 0, we get honest flow lines from the Hamiltonian H, and the A areas will become disks. This gives us a mix of flow lines and holomorphic disks. This model is called the *pearl homology* of L, where you have

- Critical Points generate the complex
- Trajectories are given by sequences of Morse trajectories and holomorphic disks.



" A string of Pearls"

When we use Novikov coefficients, we can filter the complex by energy; because we are in the monotone case, we can filter by the Maslov index. One defines the index by combining

$$\mathrm{ind}(u) = \text{Morse index of } \nabla f \text{ part} + \text{Sum of Maslov Index of Disk Part.}$$

It can be shown that the Floer index splits as

$$\partial_{\mathrm{Floer}} = \partial_0 + \partial_2 + \dots$$

where ∂_k is given by Floer trajectories of Maslov index k.
We now have a filtered chain complex, so that $CF \simeq C\mathrm{Morse}(f)$. Then there is a spectral sequence, where the E_0 page is given by the Morse homology, and converging the $HF^*(L, L)$. This is called the *Oh spectral sequence.*
What this morally means is to compute the Floer cohomology in the monotone case is to start by taking the Morse homology, and then look at higher differentials.

Example 8. *So, for S^1, we have on the first page*

$$\partial_0 \qquad p \xrightarrow{0} q$$

and on the second page

$$\partial_2 \qquad p \xleftarrow[1]{} q$$

so the spectral sequence cancels to zero.

3 The Lagrangian PSS Morphism: Ben Filippenko

The primary source for this talk is [Alb08]. The letters PSS stand for Piunikhin-Salamon-Schwarz.
Let (M,ω) be a closed symplectic manifold of dimension $2n$, and L a closed, monotone Lagrangian. Let $H_t(x)$ be a time dependent Hamiltonian, and ϕ_H be its time one flow.

Definition 8 (Minimal Maslov Number)**.** *The* minimal Maslov number *of a Lagrangian L is the positive generator of the subgroup $\mu_{Mas}(\pi_2(M,L)) \subset \mathbb{Z}$.*

Suppose additionally that L has minimal Maslov number N_L greater than or equal 2. This means that we are allowing some nontrivial holomorphic disks, but still managing to exclude the ones that will mess up our argument.

Theorem 5. *Given L and N_L as above, there are maps*

$$PSS : HF^{n-k}(L,\phi_H(L)) \to H^{n-k}(L;\mathbb{Z}/2) \mid k \le N_L - 2,$$

and

$$SSP : H^{n-k}(L) \to HF^{n-k}(L,\phi_H(L)) \mid k \ge n - (N_L - 2)$$

such that when they both are defined they are inverse isomorphism. When $N_L \ge n+2$, we get that $HF^*(L,\phi_L(L))$ is isomorphic to $H^*(L,\mathbb{Z}/2)$.

To prove this theorem, we will use a slightly different interpretation of HF^*.

- In our set-up, we will generate HF^* by chords

  $$\mathcal{P}_L(H) := \{x : [0,1] \to M \mid (*)\}$$

  where $(*)$ is the conditions

  - That the x is a hamiltonian flow
  - $x(0), x(1) \in L$
  - $[x] = 0 \in \pi_1(M,L)$.

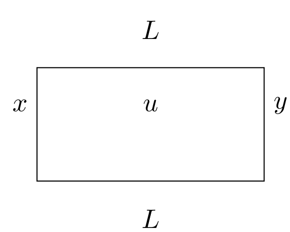
- Let J be a appropriately chosen compatible almost complex structure. Given two generators $x,y \in \mathcal{P}_L(H)$, define

  $$\hat{\mathcal{M}}_L(x,y) := \{u : \mathbb{R} \times [0,1] \to M \mid (**)\}$$

  where $(**)$ are the conditions of satisfying the hamiltonian perturbed Cauchy-Riemann equation

  $$\partial_s u + J(\partial_t u - X_H) = 0$$

  with finite energy, and boundaries defined by this diagram:

  The space $\tilde{\mathcal{M}}_L(x,y)$ comes with a natural $\mathbb{R}$ action. Define $\mathcal{M}_L(x,y)$ to be the orbits under this $\mathbb{R}$ action, and let $\mathcal{M}_L(x,y)_{[d]}$ be the union of its d-dimensional components.

- The Floer differential is defined by the count

$$\langle \partial y, x\rangle = \#\mathcal{M}_L(x,y)_{[0]}$$

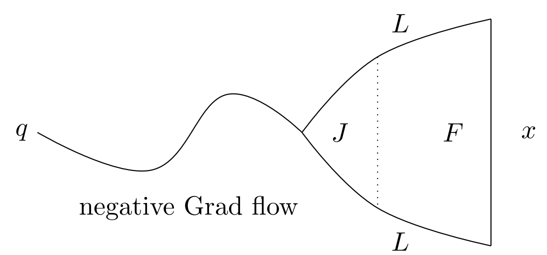
  We work over $\mathbb{Z}/2$ coefficients. The grading is given by

$$\dim \mathcal{M}_L(x,y) = \mu(x,y) := \mu_{mas}(u) \in \mathbb{Z}/N_L\mathbb{Z}$$

  Notice that our grading is only defined up to $\mathbf{Mod} N_L$. Later, we will fix $x_0 \in P_L(H)$ and set the degree of x_0 to be n, and the degree of any other element to be given by the relative Maslov index.

3.1 Construction of PSS

Fix $f : L \to \mathbb{R}$ a Morse function, and g a Riemannian metric on L such that (f,g) is Morse-Smale. Recall that the Morse complex $C_*^{morse}(L)$ is generated by $\mathrm{Crit}(f)$, the set of critical points of f. Since we know that $C_*^{morse}(L)$ is isomorphic to singular cohomology of L, we will show that $CF(L,L)$ is quasi-isomorphic to $C_*^{morse}(L)$.
This isomorphism will be constructed by counting the space $\mathcal{M}^{PSS}(q,x)$ of flows of the form:



.

Here, the labelling J means that the strip statisfies the Cauchy-Riemann equation , and F means satisfies the Hamiltonian perturbed holomorphic equation . The other labels give us the appropriate boundary conditions of the flow.
More specifically, define $\mathcal{M}^{PSS}(M,x)$ to be equations $u : \mathbb{R} \to [0,1]$ satisfying

- $\partial_s u + J\partial_t(\rho(s)X_H(t,u))$ where ρ is some smooth interpolation from 0 to 1.
- $E(u) < +\infty$
- $u(+\infty,t) = x(t)$
- $u(s,0), u(s,1) \in L$.

Each flow in this space comes with left endpoint, which defines an evaluation map on the moduli space of flows:

$$\begin{aligned} \mathrm{ev} : \mathcal{M}^{pss}(M,x) &\to L \\ u &\mapsto u(s=-\infty) \end{aligned}$$

By our setup, $\dim(M^{PSS}(M,x)) \cong n - \deg(x)\mathbf{Mod}(N_L)$.
We would like to match the space of PSS flows to the semi-infinite spaces of Morse and Floer flows. Define the semi-infinite Morse moduli space,

$$\mathcal{M}^{morse}(q,L) := \{\gamma : (-\infty,0] \to L \mid (\star)\}$$

where the condition $(\star)$ is that γ is a gradient flow line satisfying $\gamma(-\infty) = q$.
This space comes with an evaluation map $\mathrm{ev} : \mathcal{M}^{morse} \to L$ which is a homeomorphism to the unstable

manifold of q.
We can then define $\mathcal{M}^{PSS}(q,x)$ to be the fiber product over the evaluation maps of $\mathcal{M}^{Morse}(q,L)$ and $\mathcal{M}^{PSS}(M,x)$. This gives the following dimensionality result:

$$\begin{aligned}\dim(M^{PSS}(q,x)) &= \dim \mathcal{M}^{morse}(q,L) + \dim M^{pss}(M,x) - n \\ &= |q| - \deg(x)\mathbf{Mod} N_L\end{aligned}$$

Theorem 6 (Compactness Results for PSS)**.** *For $d < N_L$, $\mathcal{M}^{PSS}_{[d]}(M,x)$ admits a compactification by adding broken strips as boundary pieces.*

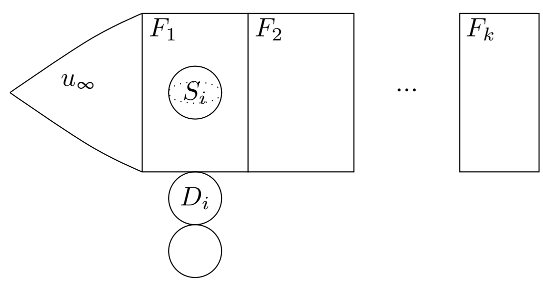
Proof. First, let's show that the requirement of energy bound exists on the holomorphic curve portions of our moduli space. Let $u, v \in \mathcal{M}^{pss}_{[d]}(M,x)$. Then

$$\begin{aligned}0 &= d - d \\ &= \operatorname{ind}(u) - \operatorname{ind}(v) \\ &= \mu_{mas}(u - v)\end{aligned}$$

Where $u - v$ is the disk that first runs from a point along u to the chord x,then along v from the chord to a different point. By monotonicity, we can relate the Maslov index to the energy by a constant λ.

$$= \lambda(E(u) - E(v))$$

We get a uniform energy bound on $\mathcal{M}^{pss}_{[d]}(M,x)$. Gromov-compactness tells us that a subsequence $(u_n) \in \mathcal{M}^{pss}(M,x)$ converges to a bubble tree of the form



where F_i are bubbling of strips, S_i are possible sphere bubbles, and D_i are possible disk bubbles trees. Since we know that the index of the components of the bubble tree add to the index of the u_n,

$$\begin{aligned}d &= \operatorname{ind}(u_n) \\ &= \operatorname{ind}(u_\infty) + \sum \operatorname{ind}(d_i) + \sum 2c_1(s_i) + \sum \operatorname{ind}(F_i)\end{aligned}$$

Since every disk bubble is a nonconstant map, we know that the indexes of the spheres and disks are greater that N_L. Our assumptions then cancel out all the N_L, as it was assumed that $d < N_L$.

$$= \operatorname{ind}(u_\infty) + \sum \operatorname{ind}(F_i).$$

We therefore forbid the bubbling of disk and sphere like components in the desired component of the moduli space. b ☐

Corollary 1. *If $d - |q| + n < N_L$, then $\mathcal{M}^{pss}_{[d]}(q,x)$ admits a compactification by broken Morse-Floer trajectories.*

If $d = 1$, then we have an identification by gluing:

$$\partial\mathcal{M}^{pss}_{[1]}(q,x) = \left(\bigcup_{p\in\mathrm{Crit}(f),|p|=|q|-1}\mathcal{M}^{morse}(q,p)\times\mathcal{M}^{pss}_{[0]}(p,x)\right)\sqcup\left(\bigcup_{y\in P_L(H)}\mathcal{M}^{pss}_{[0]}(q,y)\times\mathcal{M}_L(y,x)_{[0]}.\right)$$

where the two parts designate when these break. This includes the statements that the zero dimensional components are compact, without adding anything.

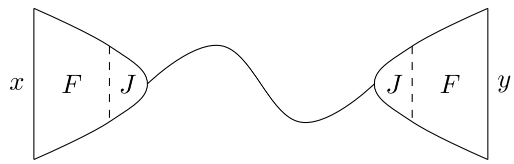
Corollary 2. *The PSS map $\langle PSS(x),q\rangle = \#\mathcal{M}^{pss}_{[0]}(q,x)$ is well defined, and is a chain map.*

The proof that this is well defined is the compactification argument before. To show that this is a chain map, we need to show that $\partial PSS(q) = PSS\partial(q)$. This equality comes from the relation

$$\langle\partial\circ PSS(q) - PSS\circ\partial(q),x\rangle = \#\partial\mathcal{M}^{pss}_{[}1](q,x).$$

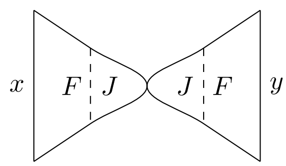
3.2 Proof that $PSS\circ PSS$ is homotopic to identity.

For $x,y\in\mathcal{P}_L(X)$, consider the moduli space $\mathcal{M}^{S^2P^2S^2}(x,y)$ of configurations that look like

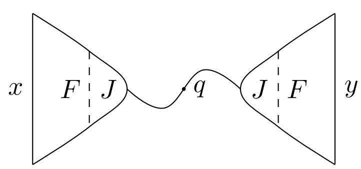


The dimension is given by $\mu(y,x)+1 \bmod N_l$. The 1-dimensional component admit a compactification with boundary with flows of these forms:

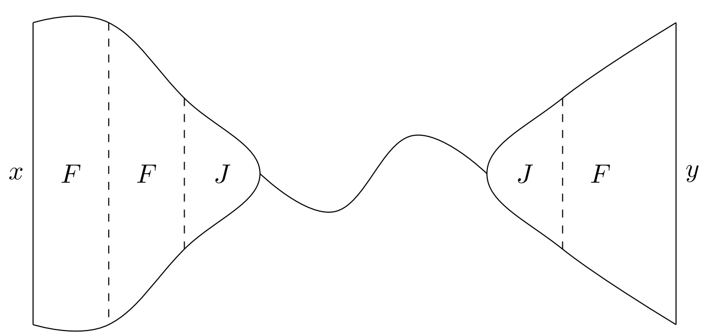
- The length of the Morse trajectory goes to zero, corresponding to a map $\eta : CF\to CF$.



- The Morse flow breaks in the middle, giving us $SSP\circ PSS$.

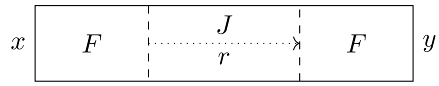


- The Floer trajectory breaks at either end: $\partial_F\circ\theta - \theta\circ\partial_F$.



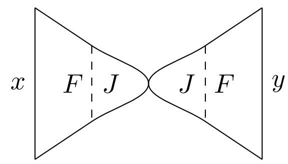
which gives us that $SSP \circ PSS - \eta$ is nullhomotopic. So we need to show that η is the identity on HF.

Proof. Consider the moduli space of things that look like



It can be compactified with boundary in several ways.

- If R goes to infinity, we get exactly the count η.



- If $R \to 0$, we get Floer trajectories from x to y. The 0 dimensional component is empty if $x \neq 0$, and consists of a single point if $x = y$, which is the constant Floer trajectory at x.

So counting these configurations gives us the identity. The other trajectories are Floer breaking on either side, which is $\partial_F \circ \tilde{\theta} - \tilde{\theta} \circ \partial_F$. But $\partial_F - \tilde{\theta} - \tilde{\theta}\partial_F = \eta - \mathrm{id}$. □

4 Products in Lagrangian Floer Theory: Morgan Weiler

There are a lot of algebraic structures on the set of intersection points on Lagrangians, and today we're going to outline them. Specifically, we will develop the products μ^k and the A^∞ relations in Floer theory. Given $k+1$ Lagrangians, the goal today is to give the maps

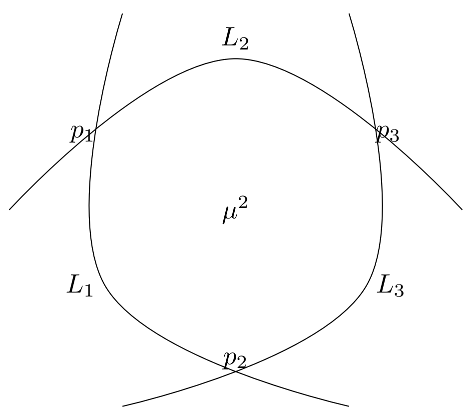
$$\mu^k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_0, L-1) \to CF(L_0, L_k)$$

which is defined by sending points $p_1, \ldots, p_k$ to the counts of holomorphic polygons with boundaries lying on the Lagrangians and corners corresponding to the intersection points.
The geometry of these maps will give us an algebra. The operation μ^1 will correspond to the Floer differential. Here's an outline for the talk:

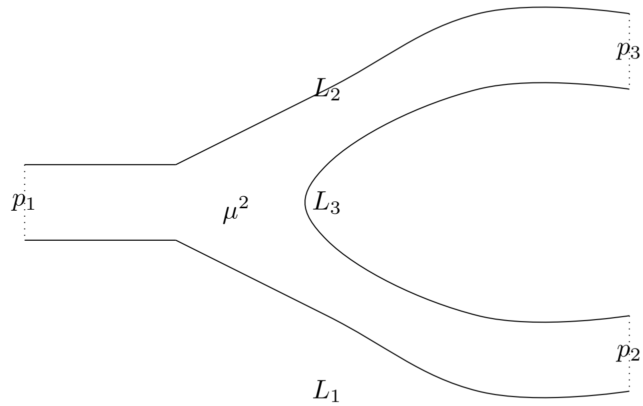
1. We will first define the product μ^2, (which we will sometimes write as a product of elements.) We will show that μ^2 satisfies the Leibniz rule with respect to the Floer differential and sketch the necessary perturbations we need to define this product.
2. We will then develop μ^k, along with the necessary perturbation data needed to define these products. We will show that these higher products satisfy the A^∞ relations, which are like a higher-dimensional Leibniz rule.
3. We will put these products together to outline the construction of the Fukaya category.

4.1 μ^2, the product

This is a map $\mu^2 : CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$. Given $p_1 \in CF(L_1, L_2)$ and $p_2 \in CF(L_0, L_1)$, the coefficient $\langle \mu^2(p_1, p_2), p_3 \rangle$ is given by counts of holomorphic disks of the following configuration:



For us, it will be useful to think of the domain of this holomorphic disk as a “disk with strip-like ends”,



with boundaries as drawn below. This can be given by the count of points in the moduli space

$$\mathcal{M}(p_1, p_2, q; [u], J)$$

which is the set of maps $u : D \to M$ such that

- u is J holomorphic with finite energy
- u has boundary conditions set up by the L_i as in the diagram
- The boundary punctures to the points p_i and q,
- u is in the homotopy class $[u]$.

Remark 1. *The automorphism group of D^2 is given by the image of 3 distinct boundary points, so we don't need to quotient this moduli-space to get unparameterized maps.*

Assume that we can define $\mathbb{Z}$ grading on the intersections. The index of a map u will be given by the Maslov index of the boundary components of the path u, combined with the canonical short paths. Given grading on the Floer theory,

$$\begin{aligned}\deg(q) - \deg(p_1) - \deg(p_2) = &\text{Maslov Index}(T_qL_0 \to T_qL_2 - \lambda_q)\\ &- \text{Maslov Index}(T_{p_1}L_0 \to T_{p_1}L_1 - \lambda_{p_1})\\ &- \text{Maslov Index}(T_{p_2}L_1 \to T_{p_1}L_2 - \lambda_{p_2})\end{aligned}$$

which (assuming some additional conditions) shows that

$$\text{ind}(u) = \deg(q) - \deg(p_1) - \deg(p_2).$$

Assuming transversality conditions for these moduli spaces, we define

$$\mu^2(p_2, p_1) = \sum_{\substack{q \in L_0 \cap L_2 \\ [u]:\text{ind}(u)=0}} \#\mathcal{M}(p_1, p_2, q; [u], J) T^{\omega[u]} q$$

We will sometimes denote this as $p_2 \cdot p_1$.

Proposition 1 (Liebniz Rule for Floer Product)**.** *If $[\omega] \cdot \pi_2(M, L_i) = 0$ for all i, then*

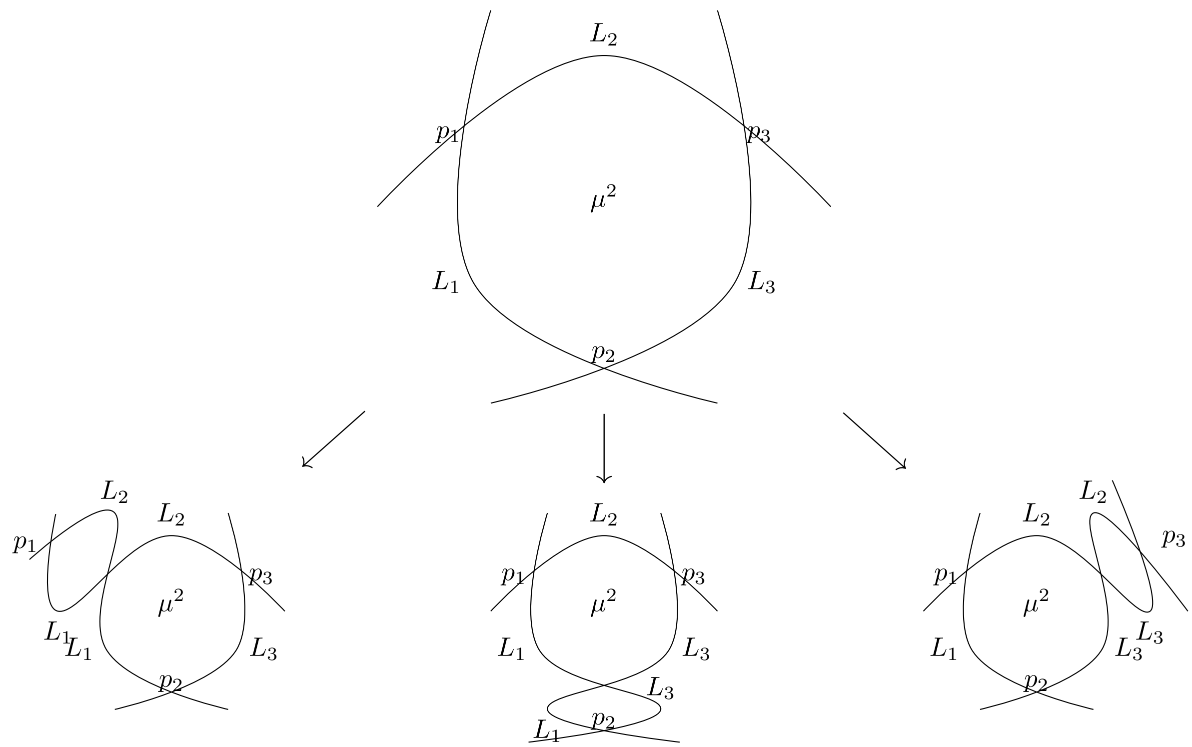
$$\partial(p_2 \cdot p_1) = \pm \partial p_2 \cdot p_1 \pm p_2 \cdot \partial p_1.$$

This means that the product is well defined on HF, and that this is associative on the level of cohomology.

Proof. We will look at the compactification of the moduli space of index 1 curves. If $\mathrm{ind}(u) = 1$, then

$$\mathcal{M}(p_1, p_2, q; [u], J)$$

is a smooth 1-manifold which admits a compactification by adding broken curves. (We don't need to worry about bubbles.)
Because of transversality, we have no curves of index less than 0, and there are no nonconstant strips of index < 1. If u_i is a family of index 1 strips going to something broken, then sum of the index of the limit components should be equal to 1. Here are the possible breakings;



These three configurations correspond to the three terms in the Leibniz rule. A gluing argument shows that boundary of $\mu(p_1, p_2, q; [u], J)$ gives us all 3 configurations. In order get this to work out, we will need to perturb these Lagrangians and pick J and H to get the same partial differential equation on all components. This is similar to setting up $HF(L, L)$, but we need to be a bit more careful. We study the PDE

$$(du - X)H \otimes \beta)^{0,1}_J = 0$$

where $\beta = dt$ near the punctures of the holomorphic disk. This is tricky, as the perturbation data needs to be close to the perturbation chosen to define the Floer complex. □

4.2 μ^k and A^∞-relations

We now define a map

$$\mu^k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_0, L - 1) \to CF(L_0, L_k)$$

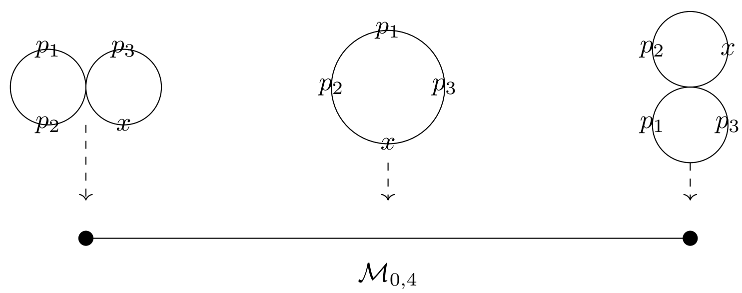
The moduli space we now consider is $\mathcal{M}(p_1, \ldots p_n, q; [u], J)$, which is the space of J-holomorphic n punctured disks $D \to M$ which extend continuously to the closed disk, with boundaries on the Lagrangians,

punctures at the intersection, in the class u and finite energy. We then quotient by the automorphism group of D^2. With the appropriate transversality condition, the dimension of the moduli space is $k+1-3+\operatorname{ind}(u)$, where the -3 comes from the action of the automorphism group.
Now, we say that

$$\mu^k(p_1, \ldots p_k) = \sum_{\substack{q \in L_0 \cap L_k \\ [u]: \operatorname{ind}(u) = 2-k}} \#\mathcal{M}(p_1, \ldots, p_k, q[u], J) T^{\omega([u])} q$$

Notice the funny looking negative index here. This will turn out to be ok, because we are parameterized by k marked points. This means that our new setup for the moduli space is the zero section of a section over the space $\mathcal{M}_{0,k+1} \times T_u C^\infty(D, M)$.
The space $M_{0,k+1}$ is compactified by the Stasheff Associahedron. Here are a couple of them.

Example 9 ($\mathcal{M}_{0,4}$)**.** *Given 4 ordered points on the boundary of the disk, there are 2 ways for it to break. First, fix the position of any 3 of the points. The final last point can drift to either the left or the right, resulting in bubbling. This gives the boundary configurations of this moduli space.*



so $\mathcal{M}_{0,4}$ is an interval.

By keeping track of bubbling at the boundary, we can show that $\mathcal{M}_{0,5}$ is a pentagon. The reason to introduce the associahedron is to notice that our degenerations of holomorphic polygons will involve the boundary strata of the associahedron.

Proposition 2 (A_∞ relations for the Fukaya Category)**.** *If $[\omega] \cdot \pi_2(M, L_i) = 0$ for all i, we get the A^∞ relations,*

$$\sum_{l=1}^{k} \sum_{j=0}^{k-1} (-1)^{\clubsuit} \mu^{k+1-l}(p_k, \ldots p_{j+l+1}, \mu^l(p_{j+l}, \ldots p_{j+1}), p_j, \ldots, p_1) = 0$$

where $\clubsuit = j + \deg(p_1) + \cdots \deg(p_j)$.

Reduction to the previous cases gives us

- $k = 1$, $\partial^2 = 0$.
- $k = 2$, Leibniz Rule
- $k = 3$, μ^2 is associative on homology.

4.3 Fukaya Category

Let (M, ω) be a nice symplectic manifold. Then we can associate a A_∞ category to this data with

- Objects given by Lagrangian submanifolds, with perturbation data,
- $\hom(L_0, L_1)$ is given by $CF(L, L_1)$.
- Composition is given by μ^2, which is associative up to homotopy.

- We have higher compositions μ^k satisfying the A_∞ relations.

Notice to set all of this up we would need to take perturbations. A choice of perturbations give us quasi-equivalent Fukaya-categories. Say that we wanted to get an actual category . Then we could work over the homology instead, but we would lose some of the data of higher product.
By dropping some of condition of "nice", we just make this category complicated. If $[\omega] \cdot \pi_2 \neq 0$, we get μ_L^0, which allows us to modify A_∞ relations.

5 Exact Triangles and Generators: Denis Auroux

This is going to be one piece of the A_∞ language we need to move on before making actual computations. While in the case of the two tori we were able to understand the Fukaya category by explicitly looking at every Lagrangian and computing, it is currently beyond our abilities to classify the Lagrangian submanifolds[3] So, we would like to determine the structure of $\mathcal{F}(X)$ by studying only certain Lagrangians. There are various challenges:

- First, how do we know when these determine the structure of the category. This is covered in this talk.
- So, when might we have enough Lagrangians to get some data on all of them. This will be covered in 6
- We will also need to know the structure of all of the higher products for a Lagrangian. We will show how to recover this from the data of finitely many products in 7

5.1 Mapping Cones/ Exact Triangles

In other categories, you might try to understand the structure of the whole category by reducing to some set of irreducible objects. For instance, if you were studying modules, you might notice that for every map between modules you have a kernel of the map, which is again a module. This process gives you a method for decomposing modules into irreducible modules.
In the Fukaya category, you have hom-sets which are given by Floer chain complexes. This is good for us geometrically, bit it is unclear how to we should construct the kernel of a morphism. While we do not have access to kernels or cokernels (and therefore cannot define exact sequences,) we can still hopefully put the structure of a triangulated category on $\mathrm{Fuk}(X)$, which will give us access to tools similar to exact sequences.

Definition 9. *Given A, B, C three graded objects, an exact triangle is a collection of morphisms between them*

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d, "g"] \\ & C \arrow[ul, "{h[1]}"] \end{tikzcd}$$

which for every object T, we get the long exact sequence

$$\cdots H^i \hom(T, A) \xrightarrow{f} H^i \hom(T, B) \xrightarrow{g} H^i \hom(T, C) \xrightarrow{h[1]} \cdots$$

One reason to hope for the structure of a triangulated category is because the B model of coherent sheaves is a triangulated category. As mirror symmetry predicts that the Fukaya category should be derived equivalent to the category of coherent sheaves, we expect some structure equivalent to triangulation to exist on the A-model side.

Example 10 (Coherent Sheaves)**.** *In the category of coherent sheaves, a short exact sequence*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

gives a long exact sequence via Ext.

Even if a category $\mathcal{C}$ is not additive, it is possible to define "triangle-like" structures on the category. We can draw intuition for what this structure should look like from topology.

[3] One piece of recent news is that we may know the Lagrangian tori in $\mathbb{R}^4$, which shows us how little we know.

Example 11 (Cones in Topology)**.** *In the category of cell complexes, given a map $f : X \to Y$, you can create a new object, the* cone *of f*

$$Cone(f) := (X \times [0,1]/X \times 0) \cup_{(x,1)\sim f(x)} Y$$

The cone of f fits into the suspension of X, which is a "shift" of cellular homology. This induces a long exact sequence on the cohomology of the spaces.

We can use this topological intuition from cones to define similar constructions in homological algebra.

Definition 10 (Cone of a complex)**.** *Given two chain complexes $M^\bullet$ and $N^\bullet$ and a chain map $f : M^\bullet \to N^\bullet$, the* cone of f *is a new chain complex* [4]

$$cone(f) := \left(M^{i+1} \oplus N^i, d = \begin{pmatrix} -d_M^{i+1} & 0 \\ -f^{i+1} & d_N^i \end{pmatrix} \right).$$

5.1.1 Triangles in A_∞ categories

We can upgrade this construction to work for *dg* modules and A_∞ categories.

Remark 2. *We drop signs from our A_∞ equations from here on!*

Definition 11. *A* differential graded module *over an dg algebra A is a*

$$A \otimes M \to M$$

with the differential interacting with multiplication by the Leibniz rule

$$d(a \cdot m) = da \cdot m \pm a \cdot dm$$

satisfying the associativity equation.
A A_∞-module *over an A_∞ algebra comes with map*

$$\mu^{k|1} : A^{\otimes k} \otimes M \to M[1-k]$$

such that for $k \geq 0$,

$$\sum \mu(a_n, \ldots, \mu(\ldots), \ldots, m) + \mu(\cdots, \mu(\cdots, m)) = 0$$

with appropriate signs.

The *dg*-relation is related A_∞ structure as $(a_1 a_2)m$ is up to homotopy $\mu^{2|1}(da_1, a_2, m) + \cdots \pm d\mu^{2|1}(a_1, a_2, m)$.

Definition 12 (Morphism of A_∞ modules)**.** *Let M, N be A_∞ modules.* morphism of A_∞ modules *is a collection of maps $f^{k|1} : A^{\otimes k} \otimes M \to N[-k]$ satisfying*

$$\sum \mu(\cdots f(\cdots)\cdots) + f(\cdots \mu(\cdots)\cdots)) = 0$$

If you have an A_∞ morphism of modules, you can construct a A_∞ mapping cones as well.

Definition 13. *Given $f \in \hom(M, N)$ an A_∞ morphism, the* cone of f *is given by*

$$Cone(f)^i = M^{i+1} \oplus N^i$$

where the multiplication maps are given by

$$\mu_C^{k|1} := \begin{pmatrix} \mu_M^{k|1} & 0 \\ f^{k|1} & \mu_N^{k|1} \end{pmatrix}.$$

One can check if f is an A_∞ module homomorphism, this construction gives a valid A_∞ structure on the cone.

[4] Your actual signs may vary, depending on who you are.

5.2 Mapping Cones in the Fukaya Category

We would like to know if the Fukaya Category has mapping cones in it. One strategy will be to enlarge the Fukaya category to its triangulated envelope, and see if we can relate the triangles in this enlargement to triangles in the original category.

- In the good case, this enlargement is not much bigger.
- In the bad case, we now have a nicer category to work with.

When we talk about homological mirror symmetry, the derived categories are really based on these enlargements, so enlarging categories does not cost us too much for doing homological mirror symmetry.

Definition 14. *A category or A_∞ category is* triangulated *if every closed morphism has a mapping cone.*

This is equivalent to saying that it sits in an exact trainable. We will look at two different constructions to add in exact triangles.

5.2.1 $\mathbf{Mod}-\mathcal{C}$ categories.

Given $\mathcal{C}$ an A_∞ category, there is category of $\mathbf{Mod}-\mathcal{C}$ (dg category of A_∞-modules) which is triangulated.

Definition 15 (Left $\mathcal{C}$ modules)**.** *Let $\mathcal{C}$ be an A_∞ category. The category of $\boldsymbol{Mod}-\mathcal{C}$, has*

- *Objects of the form $\mathcal{M} \in \boldsymbol{Mod}-\mathbb{C}$. This is a collection where for all objects $c \in \mathbb{C}$, we have $\mathcal{M}_c$ a chain complex. It also is equipped with module structure maps* $\hom(c_{k+1}, c_k) \otimes \hom(c_1, c_1) \otimes M_{c_0} \to \mathcal{M}_{C_k}[1-k]$.
- *Morphisms that we will not detail here.*

This new category is itself an A_∞ category.

There is a natural Yoneda A_∞-functor from $\mathcal{C} \to \mathbf{Mod}-\mathbb{C}$. On objects, the Yonina embedding maps $X \mapsto \mathcal{X}$, where $\mathcal{X}_c := \hom(X, c)$ for every other object c. The structure maps are $\mu^{k|1} := \mu^{k+1}$.
Similarly, there is a way to turn a A_∞ morphism into a $\mathbf{Mod}-\mathcal{C}$ morphism, and this gives you a contravariant A_∞ functor.
This is a much larger category, and the Yonina embedding is fully faithful. Fortunately, the things that we've added in are the mapping cones that we need. Unfortunately, we add in a bunch of more things.
Given $f : A \to B$ in $\mathcal{C}$ we can create an exact triangle $\mathcal{A} \leftarrow \mathcal{B} \leftarrow cone(f) \leftarrow \mathcal{A}$, in $\mathbf{Mod}-\mathcal{C}$. We say that $A \to B \to C$ is an exact triangle in $\mathcal{C}$ if the image under Yonina fits into the following triangle

$$\begin{tikzcd}
 & & C \arrow[dl] \arrow[dd, "\text{Quasi-isomorphism}"] \\
\mathcal{A} \arrow[urr] \arrow[drr] & \mathcal{B} \arrow[l, "f"'] & \\
 & & cone(f) \arrow[ul]
\end{tikzcd}$$

Usually you need to check that mapping cones satisfy a bunch of axioms. The A_∞ structure automatically gives us these axioms.

5.2.2 Twisted Complexes

Unfortunately, the $\mathbf{Mod}-\mathcal{C}$ construction is too big. There is a milder operation called *twisted complexes* which we can use to enlarge the Fukaya category.

Definition 16. *A object (E, δ_E) in the twisted category $Tw(\mathcal{C})$ is*

- *E is a formal direct sum*

$$E := \bigoplus_{i=1}^{N} E_i[k_i]$$

where $E_i \in Ob(\mathcal{C})$, and $k_i \in \mathbb{Z}$

- *The differential δ_E is a upper triangle collection $\delta_E^{ij} : E_i[k] \to E_j[k-1]$ of degree 1 maps satisfying*

$$\sum_{k \geq 1} \pm \mu^k(\delta, \cdots, \delta) = 0.$$

Example 12 (Turning Chain Complex into Twisted Complex)**.** *Given a chain complex $E_1 \xrightarrow{d_{12}} E_2 \xrightarrow{d_{23}} E_3$. We want $\mu^1(d_{12}) = 0$, and $\mu^1(d_{23}) = 0$. But we allow ourselves more flexibility by allowing $\mu^2(d_{23}, d_{12}) = \mu^1(d_{13})$. In the case of chain complexes, this ends up being zero (as $d_{13} = 0$,) but in the A_∞ case we've relaxed the condition.*

A morphism of twisted complexes is matrix $\oplus_{i,j} \hom(E_i, F_j)$. Now

$$\mu^1_{Tw}(f) = \sum \mu(\delta^F, \ldots \delta^F, f, \delta^E, \ldots, \delta^E).$$

You are a chain map if you are closed under μ^1_{Tw}. We similarly want to check μ^3_{Tw} and so on.

$$\mu^{f_k, \ldots, f_1}_{Tw} = \sum \mu(\delta, \cdots, \delta, f_k, \delta \cdots, \delta f_{k-1}, \ldots, \ldots, \delta, \ldots, \delta f_1, \delta, \ldots, \delta).$$

We can now introduce the structure of a mapping cone into the twisted complex. Suppose that $\mu^1(f) = 0$. Then we define the *cone of f* to be the twisted complex $\{A \xrightarrow{f} B\}$. So, every twisted complex is iteratively a mapping cone of some kind. This means that the category of twisted complexes is the smallest way to add in all the possible mapping cones.
The way that we detect if we have an exact triangle $A \to B \to C$ in $\mathcal{C}$ is if C is quasi-isomorphic to the mapping cone $\{A \xrightarrow{f} B\}$.

Remark 3. *There is something a bit funny about exact triangles in A_∞ categories. Assume that $\mu^2(g, f) = 0$, $\mu^2(f, h) = 0$ and $\mu^2(h, g) = 0$ in an exact triangle. Then we have maps between C and the twisted cone by*

$$C \xrightarrow{h} \{A \to B\} \xrightarrow{g} C.$$

Now, $\mu^2_{Tw} = \mu^3(g, f, h) = \mathrm{id}$, and so these maps are not just quasi-isomorphism, and are in fact isomorphism. When things compose to 0 on the chain level, then μ^3 detects that you have an exact triangle.

5.3 Generators

We say that $G_1, \cdots G_n \in ob(\mathcal{C})$ *generate* $\mathcal{C}$ if every object of $\mathcal{C}$ is quasi-isomorphic in $Tw(\mathcal{C})$ to a twisted complex built from copies of the G_i.
If we were to wok in $\mathbf{Mod}-\mathcal{C}$, then we would take iterated mapping cones. This means that we can compute all the morphisms between objects purely in terms of the generators. Then it is enough to understand

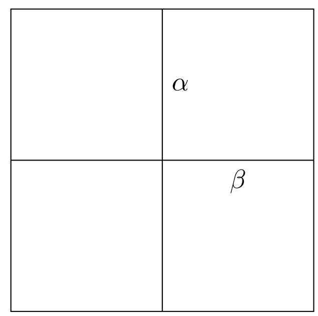
$$\mathcal{G} := \bigoplus_{i,j} \hom(G_i, G_j).$$

Unfortunately, even if you don't care about the higher products on the Fukaya category, you'll need to understand the A_∞ algebra on $\mathcal{G}$ to recover even homology of the Fukaya category. Then Yonina gives you

$$\mathcal{C} \hookrightarrow \mathbf{Mod} - \mathcal{G}.$$

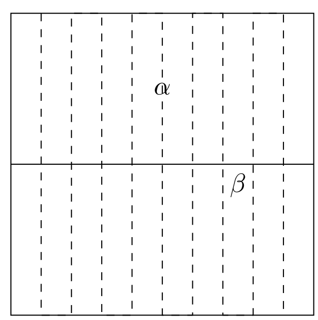
For us, we work with *split* generators. We then require that every object $\mathbb{C}$ is a direct summand in an iterated mapping cone. We get the same conclusion. Returning to our example of T^2. The longitudinal and latitudinal Lagrangians of the torus will only *split generate* $\mathrm{Fuk}(T^2)$.

T^2



In order to have a generating family the Fukaya category, we need to include all of the translates of α.

T^2



6 Lagrangian Surgery: Kuan-Ying Fang

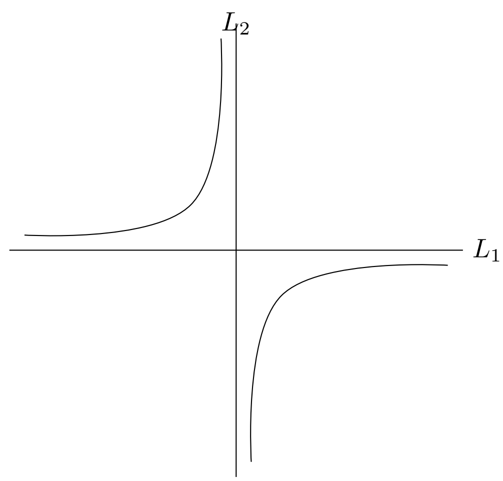
The goal of this talk is to describe Lagrangian surgery, a technique for producing the mapping cone of a morphism between two Lagrangians. Recall that a morphism between two Lagrangians is given by an intersection point of those two Lagrangians; in this talk we will be working in the simplest cases, where things intersect transversely at one point.
Let (X, ω) be a symplectic manifold, and L_1, L_2 are Lagrangians which intersect transversely at a point p_{12}. We'll locally first locally describe how we would to do Lagrangian surgery on this. In the symplectic vector space setting, the group $Sp(2n)$ acts transitively on pairs of transversely intersecting Lagrangian subspaces. We can identify the points in each Lagrangian subspace as the "real" points or the "purely imaginary" parts. Upgrading to the manifold setting, by Darboux's theorem, there is a neighborhood U containing p_{12} such sending the first Lagrangian submanifold to the real part of $\mathbb{C}^n$, and the second Lagrangian submanifold to the purely imaginary subspace of $\mathbb{R}^n$. Let

$$\begin{aligned} f_\epsilon : \mathbb{R}^n \smallsetminus \{0\} \to& \mathbb{R} \\ x \mapsto& \epsilon \log |x| \end{aligned}$$

and consider H_ϵ, the graph of df_ϵ. In $T^*(\mathbb{R}^n \smallsetminus 0\}) \simeq \mathbb{C} \smallsetminus i\mathbb{R}^n$, the submanifold H_ϵ is a Lagrangian.
Now, let's write $df\epsilon = \frac{\epsilon}{|x|^2} \sum x_i dx_i$. Then

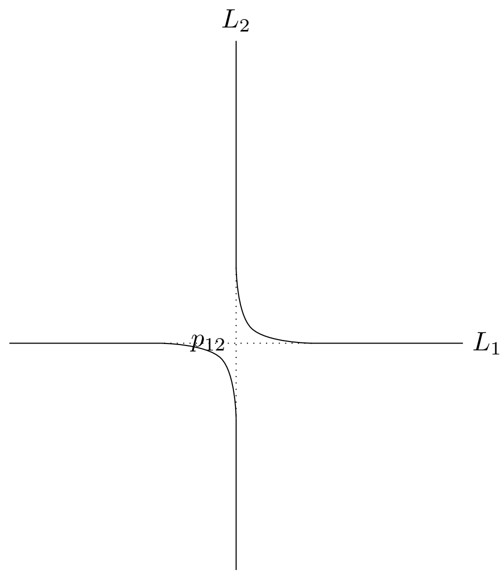
$$\begin{aligned} |x| \to 0 & \qquad H_\epsilon i\mathbb{R}^n \\ |x| \to \infty & \qquad H_\epsilon \to i\mathbb{R}^n \end{aligned}$$

This estimate shows that H_ϵ rapidly approaches the imaginary and real subspaces, which are suppose to be the images of our Lagrangians in our construction. ϵ dictates how close you are to the axis.



in local coordinates. By slightly modifying f_ϵ, we can make H_ϵ land on the real and imaginary subspace exactly. Our intuition for the lagrangian surgery of L_1 and L_2 is to modify them in these local coordinates by "swapping" in the graph H_ϵ for the intersection point. This means that we are taking an intersection point, and replacing it by breaking it into a neck. This is operation is called *Lagrangian direct sum*

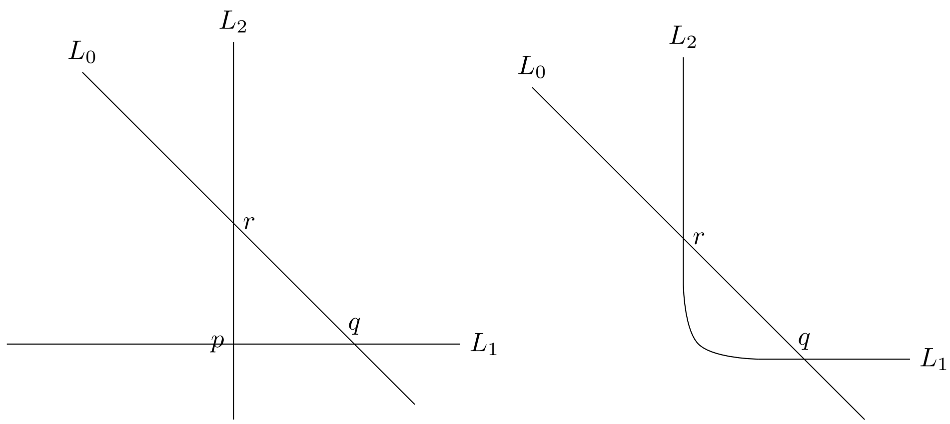
$$L_1 \#_\epsilon L_2$$



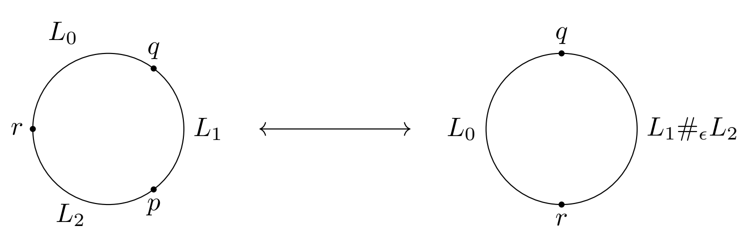
Lagrangian connect sum is a non-commutative operation in this setup. The width of the neck is determined by constant ϵ picked.

6.1 Properties of $L_1 \# L_2$.

We now have a geometric description of $L_1 \#_\epsilon L_2$, now we will be interested in what object it represents in the Fukaya category. Let L_0 be some test Lagrangian, which intersects both L_1 and L_2. Then we can draw a geometric relation between triple products of L_0, L_1, L_2 and differentials in $\hom(L_1 \#_{p12} L_2, L_0)$ in set-up below.



The intuition from [FOOO00], is that for every holomorphic disk representing a μ^3 operation with boundaries and intersections p, q and r, we have a disk between q and r.



Let's phrase this geometric intuition algebraically in order to construct a mapping cone. $\mathrm{Fuk}(X)$ is an A_∞ category, with objects L_1, L_2 and $p_{12} \in \hom(L_2, L_1)$. Since we've assumed that L_1 and L_2 intersect at only one point, we know that $\mu^1(p_{12}) = 0$. In the module category, we will build a new object, called *cone at* p_{12} (which we will denote C for this discussion.) Recall, a A_∞ module is a chain complex associated to each object of $\mathrm{Fuk}(X)$. We'll define the cone C by the relation

$$C(L_0) = \hom(L_1, L_2)[1] \oplus \hom(L_0, L_1).$$

with multiplication maps defined by

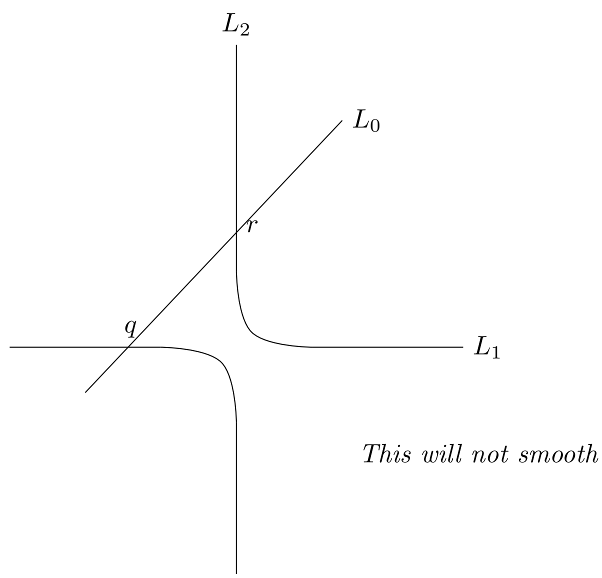
$$\mu^d_{mod}((b_2, b_1), a_{d-1}, \ldots, a_1) = \mu^d_A(b_2, a_{d-1}, \ldots, a_1), \mu^d_A(b_1, a_{d-1}, \ldots, a_1) + \mu^{d+1}_A(p_{12}, b_2, a_{d-1}, \ldots a_1)$$

For example,

$$\mu^1_{mod}(b_2, b_1) = \mu^1_A(b_2), \mu^1_A(b_1) + \mu^2(p_{12}, b_2).$$

What this means if that if you had a holomorphic strip entirely between L_0 and L_2, then the strip will still survive in the connect sum. This is why the μ^1 terms coming from L_1 and L_2 are still in this cone. However, whenever we had a μ^2 term coming from p_{12}, we will get a μ^1 contribution in the mapping cone.

Remark 4. *The order of L_1 and L_2 is important; not all holomorphic triangles will be smoothed to strips (depending on how the are oriented with the neck.)*



This is giving us an algebraic object in the module category (but we don't know that this should necessarily be the same as the object $L_1 \#_\epsilon L_2$.

Theorem 7. *In dimension $2n \geq 4$, C, the mapping cone of $p_{12} \in \hom(L_2, L_1)$, is representable by the object $L_1 \#_\epsilon L_2$ of* $\mathrm{Fuk}(X)$.

The proof of this is very technical, but we will (literally) sketch the proof out. This is the missing chapter from [FOOO00],changing Lagrangians.

Sketch of Proof.

Theorem 8 (Implicit Function theorem)**.** *Let X, Y be a Banach spaces, and $U \subset X$ open. Let $f : U \to Y$ be continuously differentiable, and let $x_0 \in U$ such that*

$$D = df(x_0) : X \to Y$$

is surjective with bounded right inverse Q. Choose δ, c two constants. Suppose that $x_1 \in X$ satisfies

$$\|f(x_1)\| < \frac{\delta}{4c} \qquad \|x_1 - x_0\| < \frac{\delta}{8}.$$

Then there exist unique $x \in X$ such that $f(x) = 0$, $x - x_1 \in 1_m Q$, and $\|x - x_0\| \leq \delta$.

We should think of c being the norm of Q.

In our case we are talking about holomorphic maps $u : \Sigma \to M$. For this situation,

$$\begin{aligned} X =& W^{1,}(\Sigma, u^* TM) \\ Y =& L^p(\Sigma, \Lambda^{0,1} \otimes u^* TM) \end{aligned}$$

Then the function we will be analyzing is

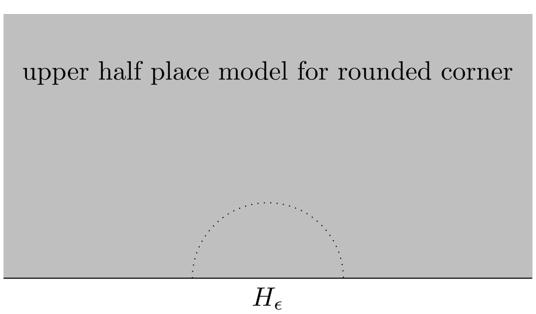
$$\begin{aligned} F_u : X \to& Y \\ \xi \mapsto& \bar{\partial}_J \exp \end{aligned}$$

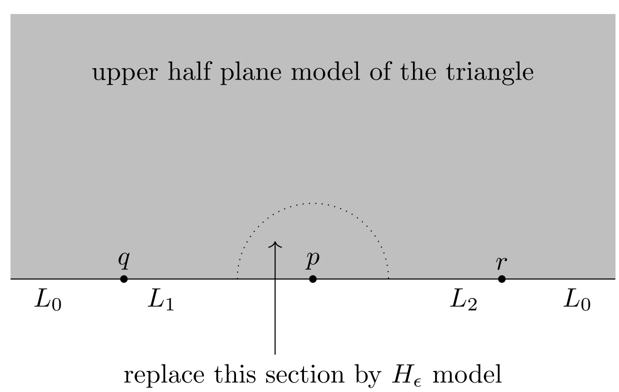
and $x_0 = 0$, and $dF_u(0) = D_u$, the linearized operator. The application of the implicit function theorem states that whenever we can find some function which makes the value of $\bar{\partial}$ small, then there is a unique thing which actually minimizes this, an actual pseudoholomorphic disk.

We'll use this to transform the disks bounded by L_0, L_1 and L_2 into disks of L_0 and $L_1 \#_\epsilon L_2$. Here are the steps that we take:

- Take the disks, and "preglue" them to things which are not pseudoholomorphic disks, but are probably very close.
- Check D_u is surjective with bounded right inverse
- $\bar{\partial}$ is small
- The implicit function theorem will then give us an honest pseudoholomorphic disk u.

So, the implicit function , in good cases, gives us way of converting things which are not pseudoholomorphic, and getting things which are pseudoholomorphic. Notice that the pregluings will *never* actually be pseudoholomorphic by analytic continuation theorems. In our case, the pregluing maps will look like this:





□

Remark 5. *Notice that the size of these disks is slightly different. Therefore, the mapping cone determined by $L_0 \#_\epsilon L_1$, really is t^ϵ related to the original mapping cone, where t is the Novikov parameter.*

7 Hochschild Cohomology and A_∞: Jeff Hicks

Here's the general strategy of what we would like to do.

- From the previous two talks, we have some hope of understanding the triangulated envelope of the Fukaya category by instead understanding the A_∞ relations on a set of generators. The plus side is that we need only understand the structure of a few Lagrangians. The downside is that understand even small products on arbitrary Lagrangians, we need to know the full A_∞ structure of the generators.
- There is a way to simplify the Fukaya category. Suppose $L_1, \dots, L_k$ is some set of Lagrangians in (X, ω), which bound a holomorphic polygon u contributing to some higher product in the Fukaya category of some surface. I can remove the count of this polygon (and in some ways simplify my Fukaya category) by putting a puncture in the surface exactly through where the polygon lived.
- More generally, given some divisor $D \subset X$, we may be able to compute $\mathrm{Fuk}(X \setminus D)$. Seidel's approach to proving Mirror symmetry outlined in [Sei02] is to show that $\mathrm{Fuk}(X \setminus D)$ and $\mathrm{Fuk}(X)$ can be related by deformations of a certain kind.
- Summarizing:

  $$\text{Simple } A_\infty \text{ category } A \xrightarrow{\text{Extend by deformation}} \text{Complicated } A_\infty \text{ category}$$

  In good cases, we can show that the deformations of algebraic objects is classified by an object called the Hochschild homology. Namely, to a *dg*-category $\mathcal{A}$, we will associate a bigraded homology theory called the *Hochschild Group*

  $$HH^k(\mathcal{A})^j$$

  and show that $A_\infty\mathcal{S}(\mathcal{A})$, the set of A_∞ structures on $\mathcal{A}$ is determined (up to homotopy) by a deformation class.

7.1 Deformation of Algebras

My notes for this section were based on [Vor].

Definition 17. *Let A be an algebra over k. A* formal deformation *of A is a $k[[t]]$ bilinear multiplication law:*

$$m_t : A[[t]] \otimes_{k[[t]]} A[[t]] \to A[[t]]$$

where $m^0(a, b)$ is the original multiplication on a, and m_t is associative. [5]

Generally, we will write the multiplication law as a power series:

$$m_t(a, b) = \sum_{k=0}^{\infty} t^k m_k(a, b).$$

Given an algebra A, we would like to know what kind of deformations it admits. One way to do this is to find deformations to kth degree, where we require associativity when we set $t^{k+1} = 0$.

Example 13. *What kind of deformations are there to first degree? Then our power series is truncated as:*

$$m_t(a, b) = m_0(a, b) + t m_1(a, b).$$

[5] This is not to be confused with the A_∞ multiplication index... yet.

Associativity of this equation is reduced to showing that

$$\begin{aligned}0 =& m_t(m_t(a,b),c) - m_t(a,m_t(b,c))\\ =& m_0(m_0(a,b)+tm_1(a,b),c)) + tm_1(m_0(a,b)+tm_1(a,b),c))\\ &- m_0(a,m_0(b,c)+tm_1(b,c)) - tm_1(a,m_0(b,c)+tm_1(b,c))\end{aligned}$$

As $t^2 = 0$ and m_0 is already associative

$$=t(m_0(m_1(a,b),c) + m_1(m_0(a,b),c) - m_0(a,m_1(b,c)) - m_1(a,m_0(b,c))$$

Satisfying this equation is enough to be a first order deformation.
One example of a first order deformation is given by derivations. Given any map $\phi : A \to A$, we can define the associated derivation as

$$m_\phi(a,b) = m_0(\phi(a),b)) - m_0(b,\phi(a)).$$

By the magic of plus and minus signs, $m_0 + tm_\phi$ is a first order deformation of A.

Notice that being a first order deformation does not in any way guarantee that you extend to an actual deformation of the algebra.

Question 1 (Motivating Question)**.** *When can we extend a first order deformation to a deformation?*

Since we have an object whose kernel is first order deformations, and the image of a different object which may be "boring" first order deformations, it is natural to set up a cohomology theory which classifies these.

Definition 18. *Let A be a k algebra. The* Hochschild complex *$C^\bullet(A,A)$ has*

- *As chain groups $C^k(A,A) := \hom(A^{\otimes n}, A)$.*
- *The differential is defined by*

$$\begin{aligned}(df)(a_0,\cdots,a_n) =& a_0 f(a_1,\dots,a_n) +\\ &\sum_{i=0}(-1)^{i+1} f(a_0,\dots,a_i a_{i+1},\dots a_n)\\ &+(-1)^{n+1} f(a_0,\dots,a_{n-1})a_n.\end{aligned}$$

The cohomology of this theory is the *Hochschild cohomology.*

Claim 1. *$H^1(A,A)$ classifies derivations on A up to inner derivations (which are given by multiplication by an element.)*

Claim 2. *$H^2(A,A)$ classifies first order deformations of A up to derivations.*

The Hochschild cohomology is actually an algebra, equipped with the *Gerstenhaber bracket*

$$\begin{aligned}[-,-] : C^m(A,A) \otimes C^n(A,A) \to& C^{m+n-1}(A,A)\\ [f,g](a_0,\cdots,a_{m+n-1}) =& \sum_k \pm f(a_0,\dots,a_k,g(a_{k+1},a_{k+n}),\dots a_n)\\ &- \sum_j \pm g(a_0,\dots,a_k,f(a_{k+1},a_{k+n}),\dots a_n)\end{aligned}$$

where I have dropped signs. Importantly, one can check that if we force f and g to commute with t, then the associativity equations become

$$[m_t, m_t] = 0.$$

and we can write

$$df = [f, m_0]$$

where m_0 is the algebra multiplication. Expanding out $[m_t, m_t] = 0$, we get the following term by term expansion:

$$\begin{aligned}[m_0, m_0] &= 0\\ 2dm_1 &= 0\\ 2dm_2 + [m_1, m_1] &= 0\end{aligned}$$

and so on. In particular, the obstruction to finding an m_2 extending the first order deformation m_1 is dependent on the exactness of $[m_1, m_1]$. This means that $H^3(A, A) = 0$, we can always extend to a second order deformation.
A send note is that our differential can be represented with the Gerstenhaber Bracket as

$$df = [m_0, f].$$

Theorem 9 (Hochschild-Extension)**.** *Suppose that $HH^k(A)$ vanishes for $k = 3$. Then $HH^2(A)$ parameterizes deformations of the algebra.*

Remark 6. *This should look suspiciously similar to language used to do things like construct deformations of complex structures by using the Kodaira Spencer map, etcetera.*

7.2 A_∞ Category

My notes for this section are based on [AAEKO13]. Recall, an A_∞ category is a collection of objects $L_i \in Ob(\mathcal{A})$ and for each pair of objects, a graded space $\mathbb{K}$-module

$$\mathcal{A}(L_i, L_j)$$

along with k-multilinear composition maps

$$m^k \in \hom_{\mathbb{K}}^{2-k}(\bigotimes_{0=1}^{k-1} \mathcal{A}(L_i, L_{i+1}), \mathcal{A}(L_k, L_0)$$

satisfying the A_∞ relations:

$$\sum_{i+j+k=l} \pm m^l(\mathrm{id}^{\otimes i} \otimes m^j \otimes \mathrm{id}^{\otimes k}) = 0$$

Here, I have not specified the range of k. If $k \geq 1$, then we get a A_∞ category. If $k \geq 0$, we get a *curved* A_∞ category.
Let us reduce to the case when $\mathcal{A}$ is just a category. There are two questions that might interest us.

- When can we add in higher morphisms to $\mathcal{A}$ making it a A_∞ category.
- What are the deformations of the category structure on $\mathcal{A}$.

For a category[6], we can define a bigraded *Hochschild complex*

$$CC^{k+l}(\mathcal{A}^l) = \hom^l\left(\bigotimes_{0=1}^{k-1} \mathcal{A}(L_i, L_{i+1}), \mathcal{A}(L_k, L_0)\right)$$

[6]You can extend this to A_∞ categories, at the cost of more associativity terms, and more signs. For a full exposition, see [Sei08]

Notice that A_∞ multiplication m^k is a $CC^2(\mathcal{A}^{2-k})$ cochain.
The differential on this cochain will be application of the composition law at every spot. Let $\phi \in CC^{k+l}(\mathcal{A}^1)$ be some cochain. Then

$$\begin{aligned} d\phi(a_1, \dots a_k) = m^2(a_1, \phi(a_2, \dots, a_k) \\ \pm \sum_{i=1}^{k-1} \phi(a_1, \dots m^2(a_i, a_{i+1}), \dots, a_k) \\ + m^2(\phi(a_1, \dots, a_{k-1}), a_k) \end{aligned}$$

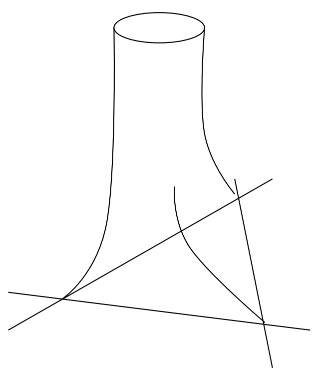
For the theory that we have set up here, there are similar deformation results as to those in the algebra case.

7.3 Geometric Interpretation of Hochschild cohomology

There is a map from symplectic cohomology to the Hochschild cohomology. In this section, we follow [Sei03b]. Let X be some symplectic manifold with nice proprieties, and pick D some divisor in X. Look at $X \setminus U$, where U is some small neighborhood of D. We can give $X \setminus U$ the structure of a symplectic manifold with contact-like boundary. In symplectic cohomology, we'll look at punctured Riemann surfaces with Reeb dynamics near the boundary. Roughly speaking, the generators are Reeb orbits, and the differential is given by counting holomorphic cylinders between those orbits.
So, how do we get a map from this to the Hochschild homology for the Fukaya category? Given an Reeb orbit o, and some set of intersections $\alpha \in \bigotimes_{i=0}^k \hom(L_i, L_{i+1})$, we define the map from Symplectic cohomology to the Hochschild complex by the count

$$\langle o, \alpha \rangle := \#\{\text{Punctured disks with boundary conditions}\}$$

of disks that look like this:



Seidel states that we should interpret these disks as counting the deformations we get by deforming the category geometrically along the divisor D.

7.4 Some Algebra and examples

Proposition 3. *Assume that $\mathcal{A}$ is a graded k-linear category, and*

$$\begin{aligned} HH^2(\mathcal{A}^j) &= 0 \text{ for } j \leq -1 \text{ and } j \neq -l \\ HH^3(\mathcal{A}^j) &= 0 \text{for } j < -l. \end{aligned}$$

Then the set of A^∞ structures (agreeing with m_1 and m_2)are exactly parameterized (up to homotopy) by deformations coming from $HH^2(A)^{-l}$.

Let's first define what a homotopy of A_∞ categories is:

Definition 19. *A* A_∞-functor *is a map* $\bar{f} : \mathcal{A} \to \mathcal{A}$ *between objects, and maps on the morphism spaces*

$$f_k : \otimes_{i=1}^{k-1} \mathcal{A}(X_i, X_{i+1}) \to \mathcal{A}(\bar{f}X_k, \bar{f}X_0)$$

satisfying the A_∞ *relations:*

$$\sum_r \sum_{i+j+k=l} \pm f_{l-j+1}(\mathrm{id}^{\otimes i} \otimes m^j \otimes \mathrm{id}^{\otimes k})$$

Two A_∞ *structures with multiplication* m *and* m' *are called* strictly homotopic *if there is another* A_∞ *functor, acting identically on objects, with* $f^1 = \mathrm{id}$.

Proof. Suppose that we would like to check that some product satisfies the A_∞ relations. We'll fix some $m^{l+2} \in HH^2(\mathcal{A}^{-l})$ which we would like to be our deformed multiplication. So, we need to fix in a whole A_∞ multiplication

$$m^2 + 0 + \cdots + m^{l+2} + m^{l+3} + \cdots$$

where we only know the first 2 non-zero terms. Let's suppose that we are trying to find m^k. The A_∞ constraint that m^k has to with relation to lower-order terms

$$dm^k = \text{An expression of } m^i \text{ for } 2 < i < k, \text{ equivalent to Gerstanharber Bracket}$$

This expression would be more complicated if we were not working in the nice case where $m^1 = 0$.

Claim 3. *This expression for this bracket is closed under the Hochschild differential.*

As the homology vanishes, we know we can always find m^k solving this equation. Inductively, we can build up the differential to solve this problem.
Now to prove the second claim of the proposition which is to show that all A_∞ structures on $\mathcal{A}$ are homotopic to one of these.
Let m' be an A_∞ structures. Let $m = m^2 + 0 + \cdots + (m')^{l+2} + m^{l+3} + \cdots$, where we've deformed the multiplication structure by m' at the first place where the Hochschild cohomology does not vanish. What would a strict homotopy have to satisfy? Since m' is suppose to be compatible with the category structure on $\mathcal{A}$, we know that $m^2 = (m')^2$, so the first order matching criteria is already filled. We are therefore looking for a collection of higher order functions satisfying the A_∞ functor relationships. This relationship can be written down as:

$$df^k = \text{Something like a Gerstanharber Bracket of } m \text{ and } m',\ f_i \text{ for } 1 < i < k$$

This bracket expression checks out to be Hochschild closed, and by the vanishing of cohomology, we can find solutions for df^k. ☐

Example 14 (The Sphere)**.** *Let's try to compute some Hochschild homology. Let's look at the sphere, and some Lagrangian on it. Since the symplectic form on the sphere is not exact, we'll need to constrain the set of Lagrangians we consider. In this scenario, we will work with the* balanced Lagrangians*, which are those which split the symplectic manifold into two parts with equal area. In this scenario, we have a well defined Fukaya category, where all of the objects are hamiltonian isotopic to the equatorial lagrangian. So we have one object,* L_1 *and the homology* $\mathcal{A}(L_1, L_1) = \{e, x\}$*, where* x *has degree* 1 *and* e *has degree* 0*.*
Let's take a look at $CC^{k+l}(\mathcal{A}^l)$*. This is maps of degree* l *from* k*-chains. Since* $\mathcal{A}(L_1, L_1)$ *only has degree* 0 *and* 1*, it means that the dimension of this space is* $\binom{k}{l} + \binom{k}{l+1}$*.*
Let's compute some Hochschild cohomology. A basis of $CC^{k+l}(\mathcal{A}^l)$ *is a string* e*'s and* x*'s of length* k*, with* l *or* $l+1$ x*'s depending on whether the string is mapped to* e *or* x*.*

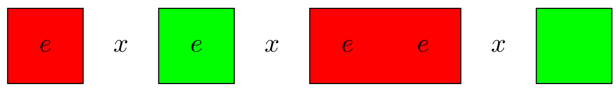
$$\underbrace{e \otimes x \otimes \cdots \otimes e}_{k \text{ things with } -l\ x's.} \quad \to e$$

$$\underbrace{e \otimes x \otimes \cdots \otimes e}_{k \text{ things with } -l+1\ x's.} \quad \to x$$

Let's call the string η. Basically, the image of η is going to be determined by inserting e's in different places. Let's look at the image of such a string η. I'm going to forget about $\pm$ signs for a moment. The image of a string is the set of all strings where either

- *An e has been inserted in a place where the are an odd number of e's written consecutively in the string (and not at the start or end of the string.)*
- *An e has been inserted in at the start or end of the string, and the string η even number of e at the start or end consecutively.*

So, given a string η, mark all the places where you can insert e's.



Notice that the insertion of an e converts an area where you are allowed to insert an e to an area where you are not allowed to insert an e; the other regions stay the same. So (forgetting about signs and cancellation) on sees that the image of applying d to any such η produces the following successive images.

$$\begin{tikzcd}
& e\otimes x\otimes e\otimes x\otimes e\otimes e\otimes e\otimes x \arrow[dl] \arrow[dr] & \\
e\otimes x\otimes e\otimes x\otimes e\otimes e\otimes e\otimes e\otimes x \arrow[dr] & & e\otimes x\otimes e\otimes x\otimes e\otimes e\otimes e\otimes x\otimes e \arrow[dl] \\
& e\otimes x\otimes e\otimes x\otimes e\otimes e\otimes e\otimes e\otimes x &
\end{tikzcd}$$

You should get a cube with vertices indexed by the valid places to insert an e. At least with characteristic 2, this means that our differential squares to zero; I'm pretty sure that the $\pm$ signs work out in general shows that differential squares to zero.
This computation shows that the Hochschild cohomology of this category is zero, and therefore the only A_∞ structure that we can put on it (up to homotopy) is the trivial one.
However, we could expand our theory to allow for deformations by allowing μ^0 deformations. By our results, these should be unobstructed.
So the short story is once we define the μ_0 obstruction term, all the other terms are defined uniquely up to homotopy.
What does this μ^0 term represent? It gives us an idea of how big the upper and lower hemisphere are of the sphere. It also parameterizes the Lagrangian, as Lagrangians on the sphere are hamiltonian isotopic up to the difference in area between their two hemispheres.

8 Genus 2 curve: Catherine Cannizzo

In [Sei11], Seidel talks both about the Fukaya category of the genus 2 surface, and the complex category which is suppose to be mirror to this. In this talk, we will only address the Fukaya category.

8.0.1 Geometric Set-up

For us, M will be the genus 2 surfaces, and $\mathcal{F}(M)$ will be a $\mathbb{Z}/2$ graded A_∞ category of *balanced curves* equipped with orientation and spin structures.

Definition 20. *Let L be a lagrangian in M, and let σ be the sphere bundle map*

$$\begin{aligned} \sigma : L \to & S(TM) \\ p \mapsto & \mathit{unit}\ T_pL \end{aligned}$$

We call L balanced *if $\int_L \sigma^*\theta = 0$, where $d\theta = \pi^*\omega$.*

Claim 4. *A nullhomollogous curve is balanced if and only if it divides M into $M_\pm$ such that*

$$\chi(M_+)/\operatorname{vol}(M_+) = \chi(M_-)/\operatorname{vol}(M_-).$$

While the balanced condition looks a little strange, notice that every lagrangian can be moved to a balanced one by displacing it by a non-zero amount of flux; while we are not picking out all of the elements of the Fukaya category, we are getting a lot of them.
Given an orientation of L, a *spin structure* on L is a lift of the SO_n-frame bundle on L to the $\operatorname{Spin}(n)$-bundle

$$0 \to \mathbb{Z}/2 \to \operatorname{Spin}(n) \to SO_n \to 0.$$

The obstruction to finding this lift is the nonvanishing of the second Stiefel-Whitney class.

8.0.2 Homological Set-up

In this section, we'll take a look at the algebraic structure we will be interested in (similar to that developed in 5.2.1.) Let $\mathcal{B}$ be a $\mathbb{Z}/2$ graded A_∞ category. Recall that one way to create triangles in a category is to consider the category of A_∞ modules over it. This construction came with a *Yoneda embedding:*

$$\mathcal{B} \to \mathbf{Mod}(\mathcal{B}) Y \mapsto \mathcal{Y} = \hom_{\mathcal{B}}(X, Y).$$

This is a full and faithful functor, and we can recover an actual triangulated category by taking cohomology. We will call this category $\operatorname{Mod}(\mathcal{B})$.

Definition 21. *A category is called* split-closed *if any endomorphism of an object which is idempotent splits that object into a direct sum of 2-objects.*
Let $\mathcal{B}$ be an A_∞ category. The split-closed derived category *of $\mathcal{B}$, is defined to be the smaller subcategory of Mod($\mathcal{B}$) such that*

- *it contains the image of the Yoneda embedding.*
- *It is triangulated*
- *It is split closed.*

We denote this category $D^\pi(\mathcal{B}.)$
We say that $\mathcal{A} \subset \mathcal{B}$ split generates *if every object of $\mathcal{B}$ is isomorphic to an element of $D^\pi(\mathcal{A})$.*

Having a set which split generates a category allows us to completely determine the structure of the split-derived category, provided that we know all of the morphisms between the generators.
In the Fukaya category, we can characterize split-generalization geometrically.

Definition 22. *Let M be two dimensional, and $L \subset M$ a lagrangian circle. A* Dehn twist *of M by L is the self-homeomorphism given by*

- *Identity outside an annulus neighborhood of L.*
- *$(s,t) \mapsto (se^{2\pi it}, t)$ in a small annulus neighborhood of L.*

In [Sei03a], it was shown that taking the Dehn twist of a L_0 along a different lagrangian L fits into an exact triangle. This is called *Seidel's exact triangle,*

$$\begin{tikzcd} HF^*(L_1, L_0) \otimes L_1 \arrow[r, "\mathrm{ev}"] & L_0 \arrow[d] \\ & \tau_{L_1}(L_0) \arrow[ul, "{[1]}"] \end{tikzcd}$$

Lemma 1. *Suppose that $\{L_1, \dots L_r\}$ are objects in $\mathcal{F}(M)$. Suppose that L_0 is isotopic (or reversed) to*

$$\tau_{L_r} \cdots, \tau_{L_1} L_0$$

Then L_0 is split-generated by L_1 through L_r.

One can also prove a result which is a little easier to check from the previous lemma:

Lemma 2. *Suppose that $\tau_{L_r} \cdots \tau_{L_1}$ is isotopic to the identity. Then $\{L_1, \dots L_r\}$ split generate the Fukaya-Category.*

8.1 Computations

We are going to compute finitely many (deformed) structure maps for a well-chosen Lagrangian. Here is our plan:

1. First we find split generators for the Fukaya category 8.1.1.
2. Even though there are multiple generators, we will able to reduce computations to one lagrangian.
3. The final picture is used for computing the structure maps μ^i for this Lagrangian.

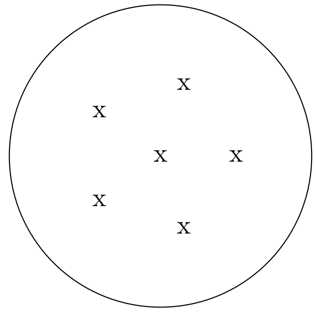
By "finite determinacy", we will not need to compute any higher order structure maps. We will then find a candidate mirror (along with B-branes), and show that these structure maps agree. This will prove our mirror statement.

8.1.1 The Split Generator

Our goal here is to exhibit the genus 2 surface as a double branched cover of $\mathbb{CP}^1$. The branching that we will use is given by $\sqrt{P(z)}$ where

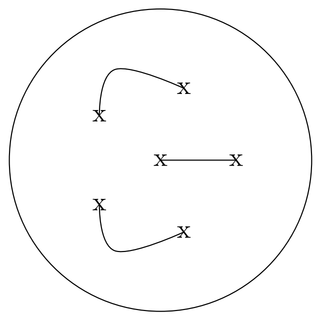
$$P(z) = z \prod_{k=0}^{5} (z - \xi_5^k)$$

where ξ_5 is a fifth root of unity. The branch that you wind up on is dependent on the parity of number of branch points a loop contains.

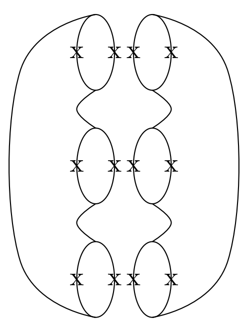


$\mathbb{CP}^1$ with 6 branch points.

If we want to visualize what the branched cover of this is, we need to select some branch curves



The topology of the branch cover is given by "opening up" the $\mathbb{CP}^1$ along these branch curves, and gluing together. The $\mathbb{CP}^1$ opened up at 3 branch curves is just a sphere with three punctures. This gives a 2:1 map from $M \to \mathbb{CP}^1$ branched at these 3 points.

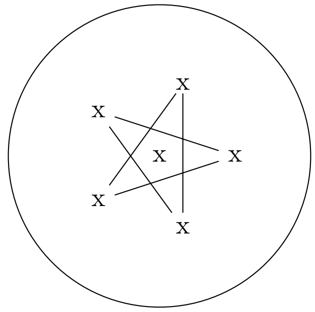


8.1.2 The Curves

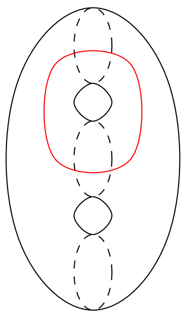
We are going to draw 5 curves on the base S^2 with endpoints on certain branches. In dimension 2, we know that every immersed closed 1-submanifold is a lagrangian.

- First, we will have to show that the curves that we describe lift up actual Lagrangians on the branch cover M
- Then we will have to argue that these curves split generate the Fukaya category.

Here are our 5 candidate curves:



We want these Lagrangians on their lift M to be invariant under "hyper-elliptic involution", which sends points on $\sqrt{P(z)}$ to $-\sqrt{P(z)}$.

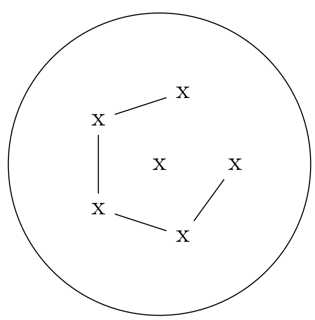


Lagrangian lifts under hyperelliptic involution

Remark 7. *Notice that a Dehn-twist along one of these lifting only corresponds to half of a twist in the base (as* $\sqrt{e^{i2\pi t}} = ei\pi t.$ *)*

8.1.3 Split Generation

The L_i have multiple intersections with each other, which means that the subgroup of the mapping class group that they generate via Dehn twists is hard to describe. Since the "easier" condition to check for generation is that a certain composition of the Dehn twists is isotopic to the identity (2), we would like to understand this group structure. To simplify, we will instead show that a different set of curves split generate, and then show that our original Lagrangians split generate these new ones. We start by looking at a different set of curves, K_1 through K_4



Since these curve only intersect each other at one point, the Dehn-twists between them satisfy the same relations as the braid groups on 5 strings. The map from the braid group to these twists is a group homomorphism

$$\begin{aligned}\Phi : \mathrm{Br}_5 \to &\text{Mapping Class Group of } M\\ \sigma_i \mapsto &\tau_{k_i}\end{aligned}$$

We can use our knowledge of the braid group to prove that these K_i split generate.

Lemma 3. *The Lagrangians $\{L_1, \dots L_5\}$ lifted to M split generate $D^\pi(\mathcal{F}(M))$.*

Sketch of Proof. The kernel of this map Φ is infinite cyclic, generated by a multiple of the diagonal element $\Delta^4 = (\sigma_1\sigma_2\sigma_3\sigma_4)^4$. One can check that the map $(\tau_{k_4}\cdots\tau_{k_1})^{10}$ is isotopic to the identity, and by 2, the K_i split generate.
A calculation shows that $\tau_{L_5}\cdots\tau_{L_1}(K_2) \simeq K_2[1]$ by isotopy, and a similar result for the other K_i. By 1 the L_i split generate the K_i, and therefore the whole Fukaya category. □

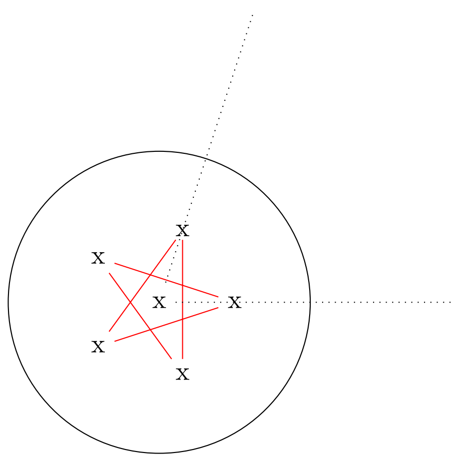
8.1.4 Reducing to 1 generator

Notice that the 5 Lagrangian L_i all look similar to each other. We have a $\mathbb{Z}/5$ action on the base S^2 interchanging the roots of unity, and it lifts to an action of M.

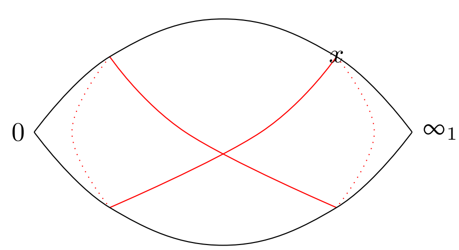
Claim 5. *$M/(\mathbb{Z}/5)$ is a orbifold sphere with 3 orbifolded points.*

The orbifolded points on the sphere are 0 and ∞. Since 0 is a branch point, this orbifolded point lifts to a single orbifolded point on M. However, ∞ is lifted to two orbifold points, which we will call $\infty_\pm$.

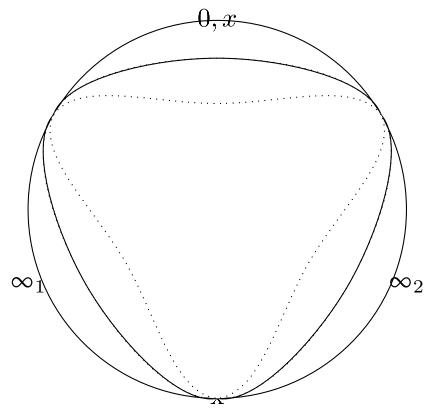
In this quotient, all 5 Lagrangians get identified to a single one. What we are going to do it to identify the two dotted lines in this picture:



The Lagrangians that survive are those contained within the "pie slice". This will look like this "orbifoldy" picture



Notice that there is still a branch point! Recall that we are suppose to "travel" through the branch point to get the $\pm\sqrt{P(x)}$ genus two curves, so we glue this together to get a sphere:



We have one lagrangian $\bar{L}$ which is no longer embedded- it has 3 self intersection points. We call this orbifolded sphere $\bar{M}$.
We want $CF^*(\bar{L}, \bar{L}) = \bigoplus_{\gamma\epsilon\Gamma} CF^*(L, L)^\gamma$ where

$$\Gamma = H_1^{orb}(\bar{M}) = (\mathbb{Z}/t)^3/\langle\Delta\rangle.$$

Here, $\Delta = (1, 1, 1)$. which acts by rotation around the orbifold points. Recall that $H_1(S^2) = 0$. Here, need to go around an orbifold point 5 times to get 0 in H_1^{orb}, or we need to go around all three points at the same time to get 0.

9 Genus 2, Part II, Catherine Cannizzo

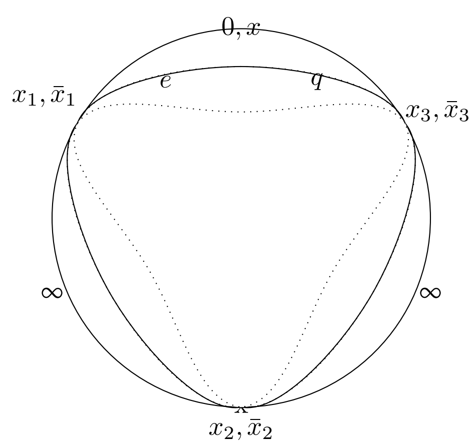
One can think of $\bar{M}$ as X/Γ, where X is the universal abelian cover $\bar{M}$. This is a 25:1 cover. In addition to indices (coming from Γ), we are going to start getting "weights," which are the elements of the group Γ which are needed to make a generator lie in $CF^*(L, L)^\gamma$.

Definition 23. *$CF^*(L, L)^\gamma = CF^*(L\gamma(L))$. In the case where $\gamma = 1$, then*

$$CF^*(L, L) = CM^*(f) = \mathbb{C}e \oplus \mathbb{C}q$$

which correspond to a min and max point of some Morse function f.

Let's do some calculations. We end up with 8 self intersections between $\bar{L}$ and a perturbation of itself.



The moduli space $\mathcal{M}(\bar{x}_0, \dots, \bar{x}_d)$ with weights γ_i is empty unless $\gamma_0 = \gamma_1 \cdots \gamma_d$. This is basically saying that if we have a disk, the boundary of the disk should be a contractible loop in the "orbifold" sense, which means that sum of the weights is zero.
Here is a table of the generators and their appropriate weights:

gen	e	x_1	x_2	x_3	$\bar{x}^1$	$\bar{x}_2$	$\bar{x}_3$	q
weight	(0,0,0)	(1,0,0)	(0,1,0)	(0,0,1)	(0,1,1)	(1,0 1)	(1,1,0)	(1,1,1)
index	0	1	1	1	2	2	2	3

The relation of the weights are that $(5,0,0) = (0,0,0)$ and $(1,1,1) = (0,0,0)$.
It turns out that the only place where μ^1 can exist is between e and q. This is due to the calculation of index of μ_k^i, where i is the number of input, and k is the count of covering of orbifold points. The degree of such a map must be $6 - 3i + 4k$. Therefore, the only time that this count can exist is when $k = 0$. The only points that have $k = 1$ are the points e and q. This is the same as counting the regular Floer differential of this equator with itself, and this Morse differential is zero. Therefore, the μ^1 map vanishes.
Now let's compute μ^2. In a deformation sense, we can expand this out as

$$\mu^2 = \mu_0^2 + \mu_1^2 + \cdots$$

The term μ_0^2 is suppose to count triangles in the pair of pants and misses the divisor: $\bar{M} \setminus \{0, \infty_1, \infty_2\}$. The term μ_1^2 counts triangles that pass through on orbifold point of $\bar{D}$, with ramification 5.

Remark 8. *There is a tensor $\eta \in \Gamma^0((T^*\bar{M})^{\otimes 2})$ which vanishes to order 2 at orbifold, which is*

$$5^3 w^2 dw^3$$

, where w is the orbifold coordinate.

To account of all of the proper index shifts due to the orbifolding :

$$\deg(\mu_k^i) = 3(2-i) + 2k \cdot 2 = 6 - 3i + 4k.$$

Claim 6. *No disks for μ^2 pass through the orbifold points.*

The counts of triangles can be broken into constant and non-constant triangles.

- There are 6 constant triangles, which are given by the triangles which satisfy $\gamma_0 = \gamma_1 + \gamma_2$ and have the appropriate indexes. Note that while these triangles are constant on the orbifolded surface, and so they get lifted to different things. After considerations involving gradient flow of the chosen Morse function, we have
  1. $\mu^2(x_i, \bar{x}_i) = -\mu^2(\bar{x}_i, x_i) = q$.
  2. $\mu^2(x_i, e) = -\mu^2(e, x_i) = x_i$.
  3. $\mu^2(\bar{x}_i, e) = \mu^2(e, \bar{x}_i) = \bar{x}_i$.
  4. $\mu^2(q, e) = -\mu^2(e, q) = q$.
  5. $\mu^2(q, q) = 0$.
- There also exist some non-constant triangles in μ_2, which you have to take a look at the paper. The general idea is that there is a large triangle on the front and back. Counting this triangle (taking into account weights, indices, etc.) gives $\mu_0^2(x_i, x_j) = \epsilon_{ijk}\bar{x}_k$.

The key point is that these relations match up with an exterior algebra,

$CF^*(L,L)$	e	x_1	x_3	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	q
$\Lambda(\mathbb{C}^3)$	1	$-\xi_1$	ξ_2	ξ_3	$x_2 \wedge \xi_3$, $\xi_1 \wedge \xi_3$	$\xi_1 \wedge \xi_2$	$\xi_1 \wedge \xi_2 \wedge \xi_3$

For μ^3, we have the following map:

$$\mu_0^3(x_3, x_2, x_1) = -e.$$

which is the triangle which goes through the front of this diagram. There are no other contributions due to degree reasons and weight restrictions.
For μ^4, we count pentagons. This map is determined by the A_∞ relations, and we don't need to compute this.
For μ^5, we do not have constant triangles.

$$\mu_1^5(x_1, \dots, x_1) = \mu_1^5(x_3, \dots x_4) = -\mu_1^5(x_2, \dots, x_2) = -e.$$

This counts a hexagon which wraps around in a really funny way.

9.0.5 How to go from here

Once you do these calculations, finite determinacy gives us a map $CF^*(\bar{L}, \bar{L}) \mapsto \Lambda^(C^3)$. We'll need the algebraic geometry side to fill in the story. The 1 in μ_1^5 says that we are counting something that goes through a single orbifold point with ramifications 5.
We can match the generators of $CF(\bar{L}, \bar{L})$ with the exterior algebra $\Lambda(V)$, where $V = \mathbb{C}^3$.
The maps μ^1 and μ^2 agree with each other, where the A_∞ multiplication turns into wedge product. However, we do not have agreement on higher orders, as the Fukaya category has nontrivial μ^3, μ^4 and μ^5. What we do is use Hochschild cohomology to deform the wedge multiplication so that it will agree with the Fukaya Category.

9.1 B-side, mirror and Category

One thing to keep in mind is that for the pair of pants, the mirror is known to be the Landau-Ginzburg model $(C^3, -v_1v_2v_3)$. So, our hope is that the mirror to the genus two curves looks something like this, as our $\mathbb{CP}^1$ missing three points is again a pair of pants.
Let V be $\mathbb{C}^3$, and let Z be the group generated by the matrix

$$\begin{pmatrix} \zeta & & \\ & \zeta & \\ & & \zeta^3 \end{pmatrix}$$

where ζ is a primitive 5th root of unity. One variety we might consider is

$$\bar{X} = V/Z,$$

which is unfortunately singular. There is a crepant resolution $\pi : X \to \bar{X}$ which is equipped with a superpotential,

$$(X, W = v_1v_2v_3 + v_1^5 + v_2^5 + v_3^5).$$

Remark 9. *Here, the technical definition of the crepant resolution is the Hilbert scheme of degree 5 of $\mathbb{C}^3$, which is ideals $I \subset \mathbb{C}[v_1, v_2, v_3]$ such that $C[v_1, v_2, v_3]/I$ has finite dimension 5. We then take a Z invariant version of this thing.*

Morally, X is like a blowup of $\bar{X}$, but it preserves the first Chern class. In this case, instead of inserting a copy of $\mathbb{CP}^2$, we get $\mathbb{CP}^2$ and the Hirzerbuch surface

$$F_3 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-3)).$$

In the toric world, $\bar{X}$ can be constructed as an affine toric variety. For a Landau-Ginzberg model on the B side, we want something that uses a superpotential function W. Normally on the B side, we would study the derived category of coherent sheaves, but in this case we will study the category of singularities of the superpotential,

$$D^\pi_{sing}(H)/$$

Here, $H = \pi^{-1}(\bar{H})$, where $\bar{H} = W^{-1}(0)/(\mathbb{Z}/5)$, which is a subset of $\bar{X}$. This is the category which we hope to be the same as $D^\pi\mathcal{F}(M)$.
The definition of the derived category of singularities is $D^\pi_{sing}(W^{-1}(0)) = D^b(W^{-1}(0))/\mathrm{Perf}(W^{-1}(0))$.

- The first portion is the bounded derived category of coherent sheaves on $W^{-1}(0)$,
- The set of perfect complexes are perfect complexes are the complex isomorphic to the bounded complexes of locally free sheaves.

The quotient that we are taking is formally inverting morphisms whose cones are perfect complexes. We do need to take the split closure here.

Claim 7. *$D^\pi_{sing}(W^{-1}(0))$ is split generated by the skyscraper sheaf at 0, denoted $S_{W^{-1}(0),0}$.*

Here is our plan of action

- We relate $D^b_{sing}(W^{-1}(0))$ with matrix factorizations on W. An example to keep in mind is that the set of points $xy = 0$ is singular because the ideal (xy) factorizes. So, the set of factorizations should tell us something about the singularities.
- From matrix factorizations $MF(W)$, we are going to need to work with $\Omega^\bullet(V)$. In our language $\Omega^\bullet(V)$ is a chain complex, and an example of matrix factorizations.
- We can apply Koszul duality to go from $\Omega(V)$ to $\Lambda(V)$.
- Applying Hochschild homology of $\Lambda(V)$ to get deformations and finite determinacy.

Definition 24. *A* matrix factorization *is $\mathbb{Z}/2$ graded projective $\mathbb{C}[V]$ module E along with a differential δ_E such that*

$$\delta_E^2 = W \cdot id_E.$$

In this case, we are meant to think of W as an obstruction to $\partial^2 = 0$.

Theorem 10 (Orlov)**.** *There is an equivalence of categories between*

$$\begin{aligned} H^0(MF(W)) &\simeq D^b_{sing}(W^{-1}(0)) \\ E &\mapsto \mathrm{coker}(\delta_e^1 : E^1 \to E^0) \end{aligned}$$

We know that $S_{W^{-1}(0),0}$ split generates the right hand side. On the left hand side, this corresponds to the object

$$E = \Omega(V)$$

with the differential $\delta_E = \iota_\eta - \gamma \wedge (-)$ where $\eta = \sum v_i \xi_i$, and $\gamma = \sum_k g_k dv_k$, and the g_k come from the superpotential,

$$\begin{aligned} g_1 &= -v_2 v_3/3 + v_1^4 \\ g_2 &= -v_1 v_3/3 + v_2^4 \\ g_3 &= -v_1 v_2/3 + v_3^4 \end{aligned}$$

A calculation shows that $\delta_E^2 = -\gamma(\eta) = W$.

9.2 Koszul Duality

In this section, we will produce an A_∞ structure on $A = \Lambda(V)$, and we'll see it's the same as the one obtained on the Fukaya category that we've already defined. We would like to go between the A_∞ structure on the category A (which has some kind of finite determinacy) to $B = \hom_{\mathbb{C}[v]}(\Omega(V), \Omega(V))$, which is related to matrix factorizations. The interchange between these two is given by Koszul duality, and a tool called the homological perturbation lemma. In any case, this allows us to work on A instead. This finite determinacy property means that up to equivalence, there exists a unique Maurer-Cartan element $\alpha \in {}^1$ such that the Hochschild-Kostant-Rosenberg map has

$$\begin{aligned}\Phi^1(\alpha_0^3) &= -v_1 v_2 v_3\\ \Phi(\alpha_1^5) &= v_1^5 + v_2^5 + v_3^5\end{aligned}$$

These notions tell us something about the deformation theory of the category. In Seidel's notation,

$$CC^d(A, A) = \prod_{i+j-1=d} \hom^j(A^{\otimes i}, A)$$

which comes with a differential and product making it a *dg* Lie-algebra[7] Given a fixed product structure, we want to see when we can deform the structure to an A^∞ structure. Given $\alpha \in CC^1$, we get a valid deformation whenever we satisfy the Maurer-Cartan equations.

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0$$

There is a classical result that states

$$\begin{aligned}HKR : HH(A, A) &\simeq \mathbb{C}[[V]] \otimes \Lambda(V)\\ HKR : \beta &\mapsto \left(\Phi\left(\beta : \xi \mapsto \sum_{j=1}^{\infty} \beta^j(\xi, \xi)\right)\right)\end{aligned}$$

which are polyvectors. There is a technical detail, in that the tangent space of the Maurer-Cartan elements consist of suitable γ of degree 0. We want to exponentiate this γ, and we will need our Lie algebra to have filtered pronilpotentcy. This can be thought of as some kind of completion. One way to do this is to add in a formal parameter $\hbar$ to our Lie algebra.
The main players are the following completed versions of $CC(A, A)$ and $C[[V]] \otimes \Lambda(V)$.

$$\mathfrak{g}^d = \prod_{i,j,k|P} \mathrm{Sym}^1(V^\vee) \otimes \Lambda^j(V)^G \hbar^k$$

and

$$\mathcal{L}^d = \prod_{i,j,k|P} \mathrm{Hom}^j(A^{\otimes i}, A)^G, h^k$$

which are quasi-isomorphic due to Kontsevich formality.

Lemma 4. *Any Maurer Cartan element $\alpha = (\alpha^0, \alpha_2) \in \mathfrak{g}^1$ such that $\alpha^0 \cong W \mod F_7\mathbb{C}([[V]])$ can be reparameterized to agree with W exactly.*

The punch line is that $\mathfrak{g}$ is easier to work with than h, and the lemma determines finite determinacy on h. The upshot is that $D^\pi(\mathcal{F}(M) = D^\pi(\mathcal{A} \rtimes Z)$ and $D^\pi_{sing}(H) = D^\pi(\mathcal{A} \rtimes Z)$.

[7] Notice the slightly different degree convention here. μ^i has degree $2 - i$ for the A_∞ structure maps, and deformations occur in CC^1.

10 Wrapped Fukaya Categories, Benjamin Gammage

So far, we have be interested in the Fukaya category of a compact manifold M. Today we will study one generalization of the Fukaya category to non-compact domains.

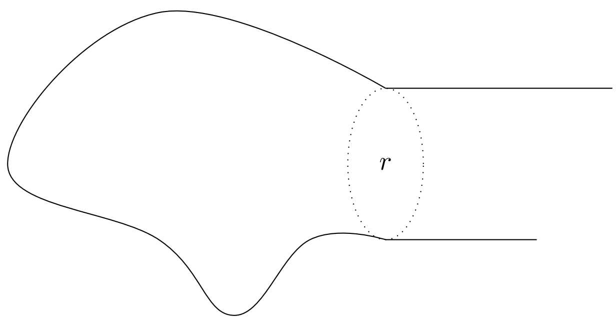
Definition 25 (Liouville)**.** *(M, λ) is called* Liouville *if $d\lambda = \omega$ is a symplectic form on M, and Z_λ generates a* complete expanding flow *on M, where*

$$\omega(Z_\lambda, -) = \lambda$$

We will think of M as having two portions

$$M = M^{in} \cup_{\partial M} [1, \infty) \times \partial M$$

The sticking out portion is called the *collar*, and is given the coordinate r



On the collar, we think of $\lambda = r(\lambda|_{\partial M})$. The boundary ∂M is *contact* with respect to the form λ.

Definition 26 (Contact)**.** *(N^{2n+1}, λ) is called* contact *if $\lambda \wedge (d\lambda)^n \neq 0$. A submanifold is called* Legendrian *if L is isotropic for α.*

In this setting, we can defined the *wrapped Fukaya Category.* For objects, we take a finite collection of exact Lagrangians such that

- $\lambda|_L = 0$ outside of a compact set.
- L can be broken into two parts, $L = L^{in} \cup_{\partial L} [0, \infty) \times \partial L$.

We will assume that $2c_1(M, L) = 0 \in H^2(M, L)$,so that we can pick bicanonical trivialization. We will assume that all L are spin, and fix spin structures. This will let us choose a $\mathbb{Z}$ grading on things like the Floer theory.

Question 2. *What type of hamiltonian perturbations are allowed as we go off towards the infinity end of the collars.*

There are two possible answers:

- Small perturbations, would give you an "infinitesimally" perturbed Fukaya category, which gives you the Nadler-Zaslow category.
- Big perturbations, which give you the wrapped Fukaya category.

Let

$$\mathcal{H}(M) = \{H : M \to \mathbb{R} \mid \text{eventuually, } H(r, y) = r^2\}$$

Since this is quadratic, the time one flow gets larger and larger the farther out you go along r. The time 1 flow of one of these Hamiltonians is the time $2r$ Reeb flow on the contact manifold $\{r\} \times \partial M$. The result is that taking a lagrangian and perturbing by one of these Lagrangians causes it to *wrap* around the collar of

M.
With this, we define

$$\mathcal{X}(L_0, L_1) := \{\text{time 1 flows starting on } L_0 \text{ and ending on } L_1\}$$

Let us assume that all the $\mathcal{X}(L_0, L_1)$ are non-degenerate.
For generic λ, this holds after a small hamiltonian perturbation of L_0 and L_1.
On a Liouville domain, there is a nice choice of almost complex structures that we should consider. Let $\mathcal{J}(M)$ be the set of almost complex structures such that

- J is ω-compatible.
- J is *contact type* on the boundary, so that $\lambda \circ J = dr$ on the ends.

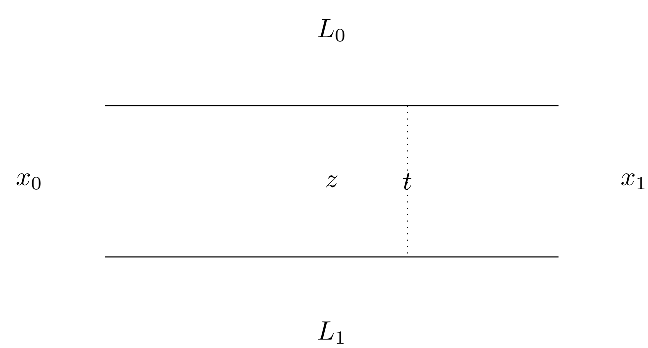
Now let $\{I_t\}$ be a family of compatible almost complex structures. Then we will take a count of holomorphic strips $u : (-\infty, \infty) \times [0, 1] \to M$ such that

- It satisfies the Floer equation;
$$(du - x_h \otimes dt)^{0,1} = 0$$
- It has boundary $u(s, 0) \in L_0$ and $u(s, 1) \in L_1$.
- The energy is defined to be $E(u) = A(x_0) - A(x_1)$.

Given $x_0, x_1 \in \mathcal{X}(L_0, L_1)$, the let $\tilde{R}(x_0, x_1)$ be the space of strips between x_0 and x_1. In the case where all of the analysis works out, we can check that

$$R(x_0, x_1) := \tilde{R}/\mathbb{R}$$

is a smooth manifold of dimension $|x_0| - |x_1| - 1$ with a compactification by broken strips $\bar{R}(x_0, x_1)$.



We don't expect $\mathcal{X}(L_0, L_1)$ to be finite, but it is ok!

Lemma 5. *For any x_1 $\bar{R}(x_0, x_1) = \varnothing$ for all but finitely many x_0.*

The idea of proof is to assign an action to the intersection points. Far away from the boundary, the action of x_0 is approximated by a term $-\int_0^1 x_0^*\lambda = \int_0^1 \lambda(x)dt = \int_0^1 2r^2 dt$, which becomes smaller as the r value of x_0 increases. This means that no x_0 too far out will contribute to the space $\bar{R}$ because it's action will be too small to contribute.

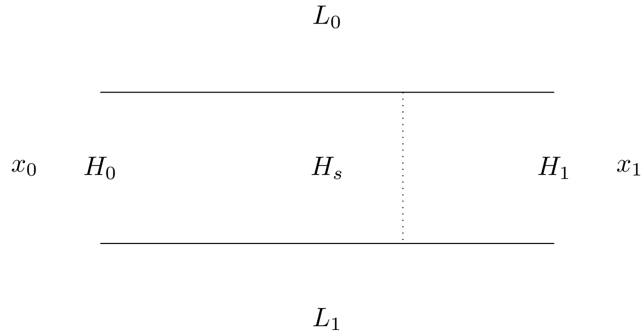
10.1 Defining the Homology Theory

As usual, let's set

$$CW^\bullet(L_0, L_1) = \bigoplus_{x \in \mathcal{X}(L_0, L_1)} |\mathcal{O}_x|$$

where $\mathcal{O}_x$ is the orientation on the line of x. We can define a differential by taking a count of $R(x_0, x_1)$. This gives us a homolog $HW^\bullet(L_0, L_1)$.

While the definition of homology goes smoothly, we will run into a problem when we try to define the higher A_∞ operations. Recall when we defined $HF^\bullet(M, H_t)$, for compact M, we showed independence of H_t by interpolating between any two Hamiltonians by a family $H_{s,t}$. We used a perturbed Floer equation which counts strips



This gives us continuation maps $HF^\bullet(M, H_0) \to HF^\bullet(M, H_1) \to HF^\bullet(M, H_0)$. To get these maps, we need a priori a bound on the energy

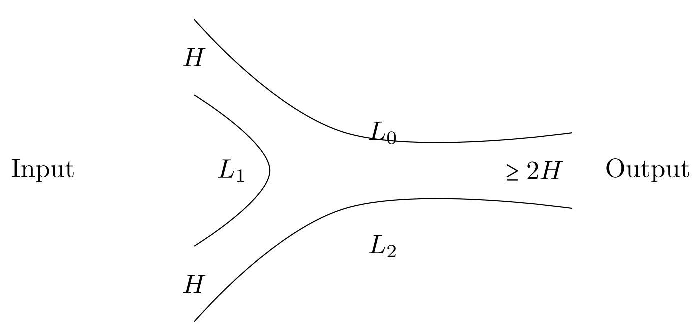
$$E(u) = A_0(x_0) - A_1(x_1) + \int_z (\partial H(u(s,t)) ds \wedge dt)$$

where the last term accounts for the change in the hamiltonian. If we can be assured that this in bounded below, we would be set in the case of the Wrapped Floer theory. For compact M, we are good, but for non-compact M, we could have H_s increase in the r direction, which would be bad!

Example 15 (What can go wrong)**.** *Let M be non-compact, and let $H_\tau = \tau r$ on the ends. We can get continuation maps from $HF^\bullet(M, H_\tau) \to HF^\bullet(M, H_{\tau'}$ for $\tau' > \tau$. However, we don't have continuation maps going the other way. This is because we are only looking at length at most τ, so we can't map the "big chords" to the "small chords."*

These two groups are *not* isomorphic. In order to get something which is invariant, we would need to take $\lim_\tau HF^\bullet(M, H_\tau) = SH_\bullet(M)$, which gives us the symplectic homology.

Let's look at disk with three punctures, with appropriate boundary conditions. To get our energy bound, there is no way to give the same perturbations everywhere, the output must have perturbation at least $2H$.



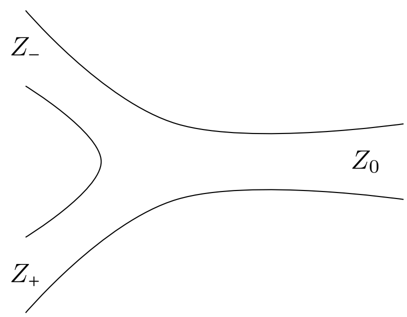
This means that the natural map on the Wrapped Floer complex is

$$CW^\bullet(L_1, L_2, H) \otimes CW^\bullet(L_0, L_1, H) \to CW^\bullet(L_1, L_2, 2H).$$

This leads to the problem, which is that the set of allowed Hamiltonians $\mathcal{H}$ are those which have $H = r^2$. Here, we need $H = 2r^2$. Luckily, we have a trick, which is that we required our Lagrangians at ∞ to do nothing. This means that if we rescale the infinite end, nothing happens. So, we will rescale infinity. Let ψ^ρ be the time $\log_\rho$ flow of Z_λ, the Liouville vector field. Since rescaling doesn't change anything, we get an isomorphism

$$CW^\bullet(L_0, L_1, H, I_t) \simeq CW^\bullet(\psi^\rho L_0, \psi^\rho L_1, \frac{H}{\rho}\psi^\rho, (\psi^\rho)^* I_t$$

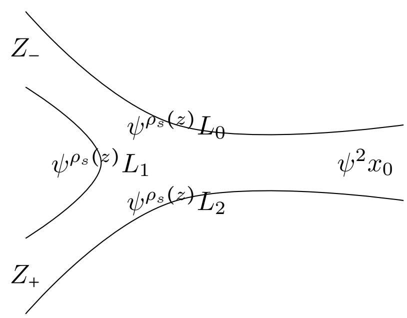
Note, that $\psi^\rho(r, y) = (\rho r, y)$. So $r^2 \circ \psi^r = \rho^2 r^2$. take $\rho = 2$, and get the isomorphism that we want. There is quite a bit of work to make this work, as we need to make this compatible with all A_∞ structures. For us, we are going to do stretching in the disk. Let $S = D \setminus \{\xi_1, \xi_2, \xi_3\}$ be a punctured disk. Take the strip-like ends, and split it into a z_0 and z_+. Take maps $\epsilon^0 : Z^- \to s$ and $\epsilon^{1,2} : Z^+ \to s$ which are assigning strip like ends to our disk.



Pick $\rho_s : \partial D \to [1, 2]$ so that it is 1 near $\{\xi^1, \xi^2\}$, and 2 near ξ_0.
Finally, choose α_s such that

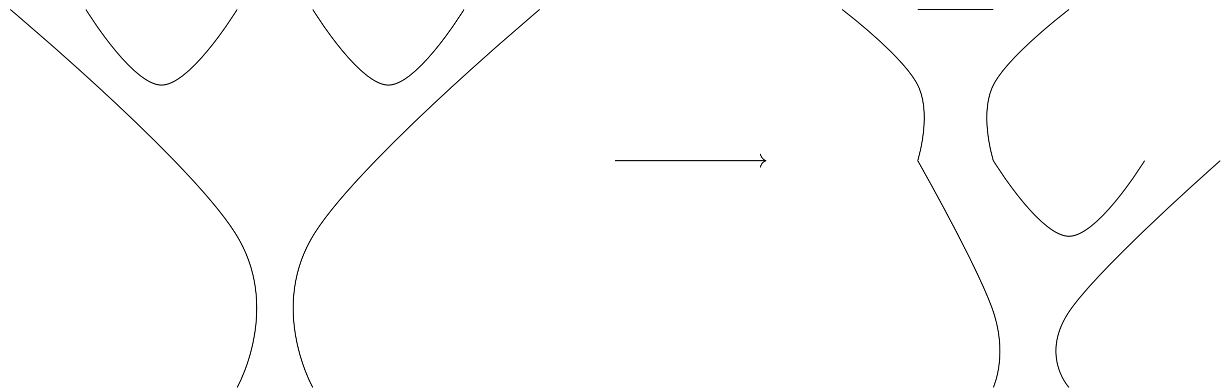
$$(\epsilon^k)^* \alpha_s \begin{cases} 2dt & k = 0 \\ dt & k = 1, 2 \end{cases}$$

Choose $H_S : S \to \mathcal{H}(M)$ and $I_s : S \to \mathcal{J}(M)$. We call this a *Floer Datum*



We'll want to make a *universal and conformal consistent* choice of Floer datum for stable disks, i.e. so that

Floer data on boundary is determined up to scaling by lower strata.

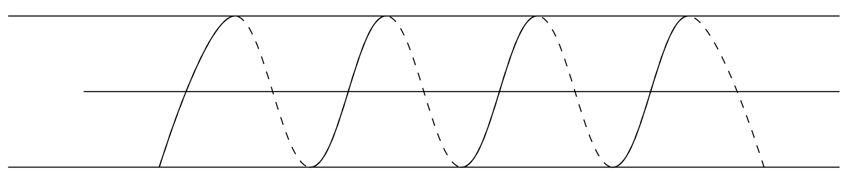


Lemma 6. *We can do this.*

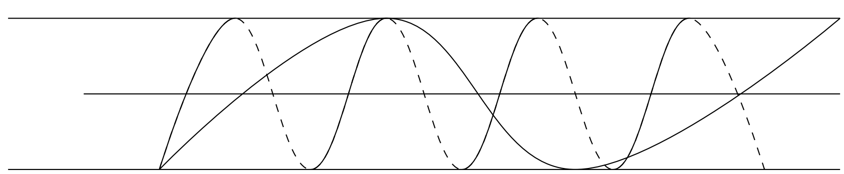
This will give us all of the μ^i, which determines the wrapped Fukaya category of M.

10.2 Example

We'll look at the cylinder T^*S^1 with the standard symplectic structure $\lambda = rd\theta$, and $H = r^2$. We let our lagrangian L to be a cotangent fiber $L = \mathbb{R} \times \{pt\}$. It turns out (spoiler alert!) that this will generate the Fukaya category.
Then $CW(L, L)$ is generated by intersection s of L with $\phi_H^1(L)$.



So $CW^\bullet(L, L) = \bigoplus_{n\in\mathbb{Z}} \mathbb{Z} \times |\mathcal{O}_{x_n}|$ Now we will calculate μ^2. This is suppose to count things which have perturbation ϕ^2 on the second input, which is to take this less squiggly lagrangian.



The take away is that $\mu^2(x_0, x_1) = x_1$. In fact, one can check that

$$mu^2(x_i, x_j) = x_{i+j}.$$

For grading reasons, there are no higher A_∞ differentials, so $HW^\bullet(L, L) = \mathbb{C}[Z]$. This will turn out to be not so surprising, as this is the set of chains on the base loop space.

11 Wrapped Fukaya Category of the Pair of Pants, Denis Auroux

12 Abouzaid's Generation Criterion, Jeff Hicks

12.1 Recall Generation Criterion

One of the problems in understanding the Fukaya category is to get an understanding of the objects in the category (as we do not have a very good understanding of what Lagrangians exist in a symplectic manifold.)

Our approach so far has been to pick out a certain class of Lagrangians, to show that these "split generate" the category. Abouzaid's generation criterion provides a way to do this.

Notation 1. *In this section:*

- (M, ω) *is a Liouville manifold*
- $\mathcal{W}$ *is the Wrapped Fukaya category of* M*.*
- $\mathcal{B}$ *is a full subcategory of* M

Theorem 11. *There is a map*

$$HH_*(\mathcal{B}) \to HH_*(\mathcal{W}) \to SH(M)$$

such that whenever the identity lies in the image, $\mathcal{B}$ *split-generates* $\mathcal{W}$*.*

Outline of Proof. There is an algebraic component and geometric component to this proof.

Algebraic Input: What we need to show is that a particular lagrangian $K \in Ob(\mathcal{W})$ is split generated by objects in $\mathcal{B}$. For any pair $L, L' \in \mathcal{B}$, there is a diagonal map

$$\Delta : HW^*(L, L') \to HW * (K, L') \otimes HW^*(L, K)$$

which is like a "dual" to the multiplication map. When we assembly these together to be a map of A_∞ modules, this is the map between

$$\Delta : \mathcal{B} \to \mathcal{Y}^l_K \otimes \mathcal{Y}^r_k$$

where these are the left and right $\mathcal{W}$-modules given by the Yoneda functor.

Whenever we have a morphism between bimodules, we get a morphism between Hochschild homology with coefficients in those bimodules. This gives us a map

$$HH_*(\Delta) : HH_*(\mathcal{B}, \mathcal{B}) \to HH_*(\mathcal{B}, \mathcal{Y}^l_K \otimes \mathcal{Y}^r_K)$$

There is an interpretation of Hochschild homology whenever the coefficients are taken in a product of a left and right module.

$$HH_*(\mathcal{B}, \mathcal{Y}^l_K \otimes \mathcal{Y}^r_k) = H^*(\mathcal{Y}^l_r \otimes_{\mathcal{B}} \mathcal{Y}^r_K)$$

This object comes with the multiplication map back down to the wrapped Floer cohomology of K, giving us the composition

$$HH_*(\mathcal{B}, \mathcal{B}) \xrightarrow{\Delta} HH^*(\mathcal{B}, \mathcal{Y}^l_K \otimes \mathcal{Y}^r_K) \simeq H^*(\mathcal{Y}^l_r \otimes_{\mathcal{B}} \mathcal{Y}^r_K) \xrightarrow{\mu} HW^*(K, K).$$

Lemma 7. *Whenever the identity of* $HW^*(K, K)$ *is in the image of this map,* $\mathcal{B}$ *split-generates* K*.*

This gives us an algebraic criterion for generation.

Geometric Input: There is a geometric interpretation of this map, which is the composition of the open-closed and closed-open maps. One interpretation of the Hochschild homology of the Fukaya category is given by the Symplectic cohomology, which is a count of pseudoholomorphic disks which have boundaries approaching Reeb orbits in the boundary.

- The Open-Closed map from the Hochschild homology to Symplectic Cohomology counts the number of punctured disks with internal puncture converging to a Reeb orbit, and boundary on the Hochschild chain.
- The Closed-Open map from the Symplectic Cohomology to the Wrapped Fukaya category counts punctured disks with one boundary point mapping to the Reeb orbit, and the other boundary point mapping to a chord.

$$HH_*(\mathcal{B},\mathcal{B}) \xrightarrow{H^*(\mathcal{OC})} SH^*(M) \xrightarrow{H^*(\mathcal{CO})} HW^*(K,K)$$

This again gives us a map from Hochschild homology of B to symplectic cohomology.

Lemma 8. *This is the same as the composition defined above.*

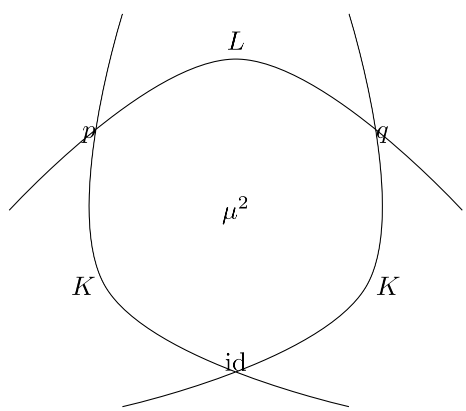
□

12.2 Algebraic Preliminaries

First, we are going to ask what algebraically we need to have a twisted complex split generate a particular Lagrangian K. We would need to show that K is a subobject of some twisted complex L. To exhibit K as a subobject of L, we need the following commutative diagram:

$$\begin{tikzcd} K \arrow[rr, "\mathrm{id}"] \arrow[dr] & & K \\ & L \arrow[ur] & \end{tikzcd}$$

If we were working with a single L, this would be the exhibition of a triangle that looks like this:

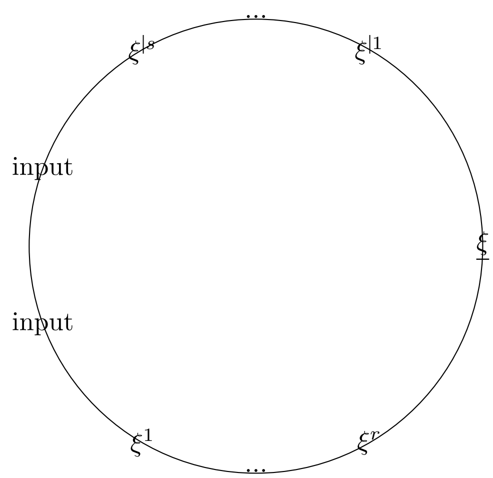


Because we are working with a twisted chain complex, the top edge can have some additional components. The existence of such morphisms is (roughly) the same as saying that $\mathrm{Hom}(K,\mathcal{B})\otimes\mathrm{Hom}(\mathcal{B},K) \to \mathrm{Hom}(K,K)$ hits the identity map. To make this precise you need the language of modules over A_∞ module, or Hochschild chains. To take into account that we are allowing the middle section to be a twisted complex, we are checking that the identity is in the image of:

$$HH^*(\mathcal{B},\mathcal{Y}^l_K \otimes Y^r_K)) \xrightarrow{\mu} HW^*(K,K).$$

Here, the Hochschild chain component gives us the twisted complex.
However, we might think that this is hard to analyze without considering K, as both the domain and codomain depend on K. There is a way to remove this dependence in at least the domain by precomposing with a "comultiplication" map from $\mathcal{B} \xrightarrow{\Delta} \mathcal{Y}^l_K \otimes Y^r_K$. Recall that a map of A_∞ bimodules also preserves some

kind of commutivity of the structure maps, so we have to give a lot of morphism. This is given by taking a count of curves with $r+s+3$ punctures, arranged as :



Where the top punctures go to chords in $\{\mathcal{X}(L_{k-1},k)\}_{k=1}^r$, the bottom punctures go to chords $\{\mathcal{X}(L_{k-1},k)\}_{k=1}^s$, and the three remaining punctures go to

- A chord between $L_{|0}$ and L_0,
- The "output" chords in $CW^*(K,L_r)$ and $L_{|s},K$.

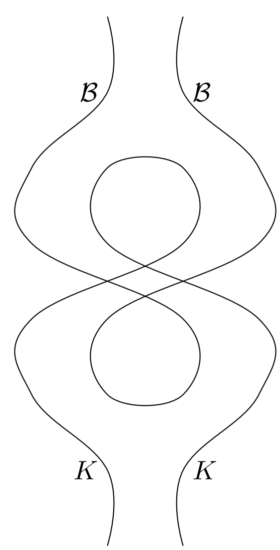
This produces a map A_∞ bi-module map

$$\Delta^{r|1|s}:\left(\bigotimes_{i=r}^{1}CW^*(L_{r-1},L_r)\right)\otimes CW^*(L_{|0},L_0)\otimes\left(\bigotimes_{k=1}^{s}CW^*(L_{|k},L_{|k-1})\right)\to CW^*(L_{|s},K)\otimes CW^*(K,L_r).$$

To actually define this map takes a substantial amount of work, to check that the moduli spaces that are in this count are well defined.
Putting this together gives us a composition

$$HH_*(\mathcal{B},\mathcal{B})\xrightarrow{\Delta}HH^*(\mathcal{B},\mathcal{Y}^l_K\otimes\mathcal{Y}^r_K)\simeq H^*(\mathcal{Y}^l_K\otimes_{\mathcal{B}}\mathcal{Y}^r_K)\xrightarrow{\mu}HW^*(K,K).$$

This composition amounts to taking a count of holomorphic disks that look like this:



We will want to express these disks geometrically.

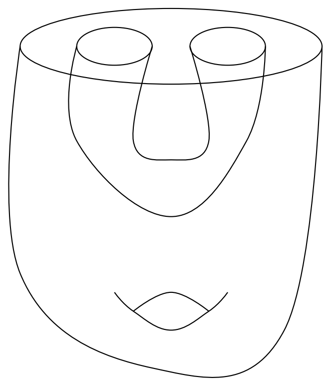
12.3 Symplectic cohomology

This can be expressed with symplectic cohomology. We give a quick overview here of the definition:
A *Reeb vector field* R_α on a contact manifold (Y, λ) such that

$$\iota_{R_\alpha} d\alpha = 0$$
$$\alpha(R_\alpha) = 1$$

- The generators are Reeb orbits in the contact boundary
- The differential is given by taking a count of holomorphic disks going between Reeb orbits

The picture you should have in mind is something like this:



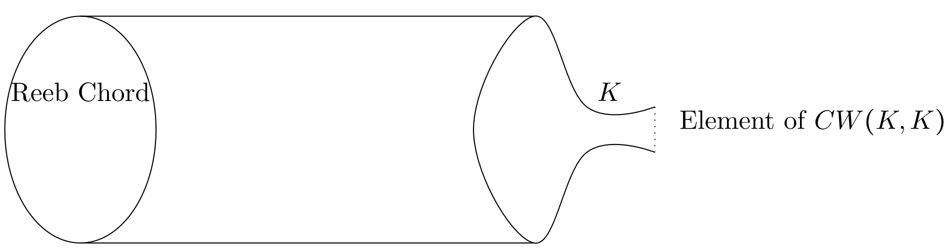
The differential counts the cylinders between these orbits. The moduli space cylinders satisfies a compactification by broken curves causing the differential to square to zero.

12.3.1 Identity

The identity component in symplectic homology is given by the Reeb orbits which bound holomorphic disks.

12.3.2 Open Closed and Closed Open maps

There is a map from symplectic cohomology to the Lagrangian cohomology which is given by counting disks with a single interior puncture and a single marked point on the boundary. The map is given by counting rigid holomorphic disks with the strip-like end going to the self Reeb chord of a Lagrangian K.

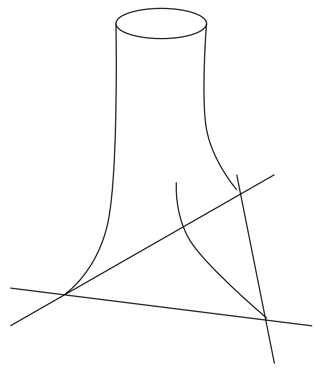


We call this the "closed-open map" $CO : SH(X) \to HW(K, K)$.

Claim 8. *The identity of $HW(K, K)$ is in the image of the open-closed map.*

Proof. The image of Reeb orbits which bound holomorphic disks is the identity in the Wrapped Fukaya category. These are cylinders attached to disks by a gluing argument. ☐

There is also a map from the Hochschild Homology of the Wrapped Fukaya category to the Symplectic Cohomology, given by counting disks of the following configuration:



This map gives us a "geometric deformation" of the Fukaya category by considering the disks which are removed by going through this divisor.
The space of open closed and closed open cylinders can be glued together into the space of all cylinders from cycles of Lagrangians in $\mathcal{B}$ to

12.4 Putting it Together

References

[AAEKO13] Mohammed Abouzaid, Denis Auroux, Alexander Efimov, Ludmil Katzarkov, and Dmitri Orlov. "Homological mirror symmetry for punctured spheres". In: *Journal of the American Mathematical Society* 26.4 (2013), pp. 1051–1083.

[Alb08] Peter Albers. "A Lagrangian Piunikhin-Salamon-Schwarz morphism and two comparison homomorphisms in Floer homology". In: *International Mathematics Research Notices* 2008 (2008), rnm134.

[AS+10] Mohammed Abouzaid, Ivan Smith, et al. "Homological mirror symmetry for the 4-torus". In: *Duke Mathematical Journal* 152.3 (2010), pp. 373–440.

[Aur14] Denis Auroux. "A beginners introduction to Fukaya categories". In: *Contact and Symplectic Topology*. Springer, 2014, pp. 85–136.

[CXGP91] Philip Candelas, C Xenia, Paul S Green, and Linda Parkes. "A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory". In: *Nuclear Physics B* 359.1 (1991), pp. 21–74.

[Flo88] Andreas Floer. "Morse theory for Lagrangian intersections". In: *Journal of differential geometry* 28.3 (1988), pp. 513–547.

[FOOO00] K Fukaya, YG Oh, H Ohta, and K Ono. "Lagrangian intersection Floer homology-anomaly and obstruction". In: *preprint* (2000).

[Giv99] Alexander Givental. "The mirror formula for quintic threefolds". In: (1999).

[Kon94] Maxim Kontsevich. "Homological algebra of mirror symmetry". In: *arXiv preprint alg-geom/9411018* (1994).

[KS00] Maximilian Kreuzer and Harald Skarke. "Complete classification of reflexive polyhedra in four dimensions". In: *arXiv preprint hep-th/0002240* (2000).

[KS01] Maxim Kontsevich and Yan Soibelman. *Homological mirror symmetry and torus fibrations.* World Scientific, 2001.

[LLY97] Bong Lian, Kefeng Liu, and Shing-Tung Yau. "Mirror principle I". In: *arXiv preprint alg-geom/9712011* (1997).

[PZ98] Alexander Polishchuk and Eric Zaslow. "Categorical mirror symmetry: the elliptic curve". In: *arXiv preprint math/9801119* (1998).

[Sei02] Paul Seidel. "Fukaya categories and deformations". In: *arXiv preprint math/0206155* (2002).

[Sei03a] Paul Seidel. "A long exact sequence for symplectic Floer cohomology". In: *Topology* 42.5 (2003), pp. 1003–1063.

[Sei03b] Paul Seidel. "Homological mirror symmetry for the quartic surface". In: *arXiv preprint math.SG/0310414* 186 (2003).

[Sei08] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory.* European Mathematical Society, 2008.

[Sei11] Paul Seidel. "Homological Mirror Symmetry for the genus two curve". In: *Journal of Algebraic Geometry* 20.4 (2011), pp. 727–769.

[She11] Nicholas Sheridan. "Homological mirror symmetry for Calabi-Yau hypersurfaces in projective space". In: *arXiv preprint arXiv:1111.0632* (2011).

[Vor] Alexander A Voronov. "LECTURE 16: DEFORMATION THEORY OF ASSOCIATIVE ALGEBRAS". In: ().