0.1 The Space of Solutions, 6.6

So, we are intersted in studying solutions to the Floer Equation:

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0$$

where $u : \mathbb{R} \to \mathcal{L}W$ is suppose to represent a flow line of the action functional. We want to know what the structure of this space is. In particular,

- Is it compact?
- Does it admit a nice compactification?

0.1.1 Energy

To every map $u : \mathbb{R} \to \mathcal{L}W$ we can associate an energy. Taking "energy" to roughly mean "the integrated change of the functional" we define the function to be

$$E(u) := -\int_{-\infty}^{+\infty} \frac{d}{ds}\mathcal{A}_H(u(s))ds$$

As the change in the $\mathcal{A}_H(u)$ with respect to s is the negative length of the gradient, as u is a flowline.

$$E(u) = -\int_{-\infty}^{+\infty} -\|\operatorname{grad} A_H\|^2 ds$$

Recall, length on the loop space comes from some integral on the manifold

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\left(\int_{S^1}\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t} - X_t(u)\right|^2 dt\right)ds$$

Using that u is a solution

$$= \int_{-\infty}^{+\infty}\left(\int_{S^1}\left|\frac{\partial u}{\partial s}\right|^2 dt\right)ds$$

So, this energy function has the following really nice properties:

- The energy is positive–in fact, it is clearly the area of the holomorphic strip represented by u.
- The energy is the difference in the action function al between critical points that.
- The energy of a solution is zero if and only if that solution is zero if andonly if $\frac{\partial u}{\partial s} = 0$, which is if and only if u is a critical point of the action functional.

0.1.2 The Moduli Space of Solutions

Consider now the space

$$\mathcal{M} = \{u : \mathbb{R} \times S^1 \to W \mid uu \text{ is contractible smooth solution with finite energy}$$

Here are a number of properties that we will want to prove about the Moduli space of solutions:

1. **Geometric Interpretation** We can identify this space with the space of cylinders which tend toward periodic orbits.
2. **Gromov Compactness:** Suppose that every sphere has 0 energy, and that W is compact. Then the space $\mathcal{M}$ is compact in the $C^\infty_{loc}(\mathbb{R} \times S^1, W)$ topology. (This is not the standard Gromov compactness. Notice for instance that breaking flowlines converge to orbits or flow lines in this topology).

3. **Elliptic Regularity** Every C^1 solution is of class C^∞. The topologies C^k_{loc} coincide for all $k \in \mathbb{N}, \infty$.
4. **Index and smoothness:** Let $x_\pm$ be two different orbits (critical points of the action), and let h be a pertubation of the Hamiltonian. Then let $\mathcal{M}(x_-, x_+)$ be those flows which tend toward $x_\pm$ with this pertubation is a manifold, whose dimension is dependent only geometric properties of $x_\pm$.

This encapsulates the majority of work that we need to understand the moduli space of solutions, which is a primary component of the definition of Hamiltonian Floer Homology. So, let's get started.

0.2 Gromov Compactness

This is a statement about the moduli space of things like J-holomorphic curves in general, and there are many variations of the Gromov compactness theorem based on the choice of Moduli space that you have chosen. In this cas, we will be making a statement about the compactness of the moduli space of cylinders. We will use the bound of finite energy to force this compactness. One should note that in this section it is critical to have the assumption that $\int_{[S^1]} \omega = 0$ for all spheres, to prevent a phenomenon called "sphere bubbling."
Assume that u_n is a sequence of maps.

- We will show that the energy bound forces this to be an equicontinuous family. By Arzelà Ascoli, there is a subsequence that converges uniformly (in the C^0_{loc} topology). This gives us a candidate element for convergence c_0– however, we do not know that it is necessarily smooth.
- Some elliptic regularity argument will be needed to show that this solution smooth.
- Another elliptic regularity argument will show that the C^0_{loc} and C^∞_{loc} topologives coincide on $\mathcal{M}$.
- This will show that $\mathcal{M}$ is compact.

So, this will show that our space is compact. The geometrically interesting part of the proof will be this equicontinuous argument.

Proposition 0.2.1 There exists a constant bounding

$$\|\operatorname{grad}_{(s,t)} u\|^2 = \left\|\frac{\partial u}{\partial s}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2$$

which is independent of s, t or u.

What is the idea of the proof? The concentration of this value to a small section of our curves will cause th formation of a sphere in the space W with positive energy, which is forbidden.
We can reparameterize the domain by translation to make the assumtion that $s_k, t_k = 0$ for all of these. We can further reparameterize by scaling the domain to create function v_k which has gradient of length 1 at the origin. Call this scaling parameter R_k. These have the same energy, but they satisfy a slightly modified version of the Floer eqution:

$$\frac{\partial v_k}{\partial s} + J(v_k)\frac{\partial v_k}{\partial t} + \frac{1}{R_k}\operatorname{grad} H = 0$$

In other words, zooming in on the domain reduces the effects coming form the Hamiltonian deformation of the J-holomorphic solution. We should expect the limit of the v_k, if it exists, to be a J-holomorphic sphere. Let ϵ_k be a parameter so that $\epsilon_k R_k \to \infty$, even though $\epsilon_k \to 0$. The idea is that $\epsilon_k R_k$ is going to represent a neighborhood going to infinity in these new coordinates, yet not fast enough to catch up with many off the effects from the old coordinates on u_k.

Through an arguement which does not relate to the geometry, you can choose your initial points and a sequence $\epsilon_k \to 0$ satisfying these bounds on a ball of radius $B(0, \epsilon_k R_k)$.

$$\|\operatorname{grad}_{(s,t)} v_k\| \leq 2$$

where $r_k \to \infty$. If it were the case that our domain were compact, an obvious candidate for s_k, t_k initially would have been the maximum value of the function. This says that up to some rescaling and working locally, you can still get this.

Now we have a set of functions v_k which have bounded gradient, and therefore form an equicontinuous family. By elliptic regularity and Azerlà Ascoli, there is a convergent subsequence which converges to a smooth solution. Let's call this solution v. By our clever reparameterization, we have that this thing satisfies the "limiting bad behaviors of v_k around $0, 0$. "

- $\|\operatorname{grad}_{0,0} v\| = 1$. In particular, this is not constant function.
- $\|\operatorname{grad}_{s,t} v\| \leq 2$. This means that the function doesn't change too rapidly.
- The function is J -holomorphic.

While our reparameterizations created an interesting new function, they did not destroy many of the energy properties.

- v has a finite amount of energy.
- The symmplectic area of v is finite and non-zero.

Let's prove these items.

Lemma 0.2.2 v has a finite amount of energy.

Proof. The idea is that we look at balls coviner $\mathbb{R}^2$ at a rate of $\epsilon_k R_k$. In our original scaling, these are balls that of size ϵ and therefore going to 0. We know that the energy of a curve comes in rougly two components: the gradient portion of it, and the hamiltonian contribution. We know that scaling should lower the hamiltonian contributions, and we know that the gradient is bounded.

We take the energy over larger and larger balls covering $\mathbb{R}^2$. Define $B_k = B(0, \epsilon_k)$. Then

$$\begin{aligned}
\int_{B(0,\epsilon_k R_k)} \|\operatorname{grad} v_k\|^2 &= \int_{B(0,\epsilon_k)} \left(\left\| \frac{\partial v_k}{\partial s} \right\|^2 + \left\| \frac{\partial u_k}{\partial t} \right\|^2 \right) dtds \\
&= \int \left(\left\| \frac{\partial v_k}{\partial s} \right\|^2 + \left\| \frac{\partial u_k}{\partial t} - X_t(u_k) + X_t(u_k) \right\|^2 \right) dtds \\
&\leq \int \left(\left\| \frac{\partial v_k}{\partial s} \right\|^2 + 2 \left\| \frac{\partial u_k}{\partial t} - X_t(u_k) \right\|^2 + 2\|X_t(u_k)\|^2 \right) dtds \\
&\leq 3E(v_k) + 2 \int_{B_k} \|X_t\|^2 dtds \\
&\leq 4E
\end{aligned}$$

Ok, the last integral tends to zero as the ball B_k is smaller and smaller. If we can show that the energy is bounded, which we will do next time, we will be done.
Since v is J-holomorphic, it's energy is just given by $\int \|\operatorname{grad} v\|^2$. By Fatou's Lemma, we are done. ∎

Lemma 0.2.3 v has finite nonzero symplectic area.

Proof. In the J-holomorphic case, the symplectic area is less than 2 times the energy.

$$\begin{aligned}
\int_{R^2} v^* w &= \int_{R^2} \omega \left(\frac{\partial v}{\partial s}, \frac{\partial v}{\partial t} \right) dsdt \\
&= \int \|\frac{\partial v}{\partial t}\|^2 dsdt \leq 2E(v).
\end{aligned}$$

∎

Here, we depart from Audin-Damien. So far, v gives us a finite energy map from $\mathbb{CP}^2 \setminus \{\infty\}$ to the manifold W. There is a removable singularity theorem for J holomorphic curves with finite energy .This gives a map from the sphere to W called a "bubble," which is an obstruction to compactness.

However, our assumption is that there are no holomorphic spheres of positive area. So, we should be really upset that we now have this sphere. So, we should have been compact.

Remark. In the case where you can restrict bad behavior of these spheres, you can still get a kind of compactness, but it may be at the cost of working with "bubble trees".

0.3 Geometric Interpretation

In this section, we prove the following Theorem:

Theorem 0.3.1 Geometric Interpretation of the Floer Space

Suppose that all periodic trajectories of X_t are nondegenerate; then for ever $u \in \mathcal{M}$, there exist two critical points $x_\pm$ of $\mathcal{A}_H$ such that

$$\lim_{s \to \pm\infty} u(s, \cdot) = x_\pm$$

in the $C^\infty(S^1, W)$ topology. Moreover,

$$\lim_{s \to \pm\infty} \frac{\partial u}{\partial s}(s, t) = 0$$

uniformly in t.

Roughly, this says that the Floer space behaves in many of the same ways that we expect Morse flow spaces to behave. Here is the outline of the proof:

1. First, we prove that there is a sequence of s_k such thatu_{s_k} which converges to a critical point, with the action of this limit matching the limit of the action (independent of the sequence s_k chosen. In particular, we show that $\lim_{s \to \infty} \mathcal{A}_H(u_s)$ exists.
   (a) There exists a sequence s_k such that u_{s_k} converges in the $C^0(S^1, W)$ topology to a limit x
   (b) This limit x is a smooth and a critical point of $\mathcal{A}_H$
   (c) The action functional has the limiting value on this sequence.
2. Nondegeneracy gives us a finite number of periodic tragectories.
3. Show that for every $s_k \to \infty$, there is a subsequence $s_{k'}$ that with the property that $u(s_{k'}) \to x$, some orbit.
4. Finishing Details.

So, with this guideline, let's prove the theorem.

Proposition 0.3.2 Let $u \in \mathcal{M}$. There exist critical points $x_\pm$ such that $\lim_{s \to \pm\infty} \mathcal{A}_H(u_s) = \mathcal{A}_H(x_\pm)$.

Lemma 0.3.3 There exists a sequence s_k such that u_{s_k} converges in the $C^0(S^1, W)$ topology to a limit u_+.

Proof. Because the cylinder has finite energy, we have that

$$\int_{\mathbb{R}} \left(\int_{S^1} \left| \frac{\partial u}{\partial t} - X_t(u) \right|_J^2 dt \right) ds < \infty$$

we have that the L_2 norm of $\frac{\partial u_s}{\partial t} - X_t(u_s)$ integrates to zero along s. Therefore there is a sequence of s_k such that

$$\left\| \frac{\partial u_{s_k}}{\partial t} - X_t(u_{s_k}) \right\|_{L^2,t,J} \to 0$$

While the norm here is defined using J, by compactness, we can switch to some simpler norm and not change this limit value. Let's use the Euclidean norm from an embedding of W into $\mathbb{R}^m$. This gives us the limit

$$\|\dot{u}_{s_k} - X_t(u_{s_k})\|_{L^2(S^1,\mathbb{R}^m)} \to 0$$

By compactness, $X_t(u(s_k, t))$ is bounded. Therefore $\|\dot{u}_{s_k}\|_{L^2}$ is bounded. Therefore u_{s_k} is an equicontinuous family. By the Azerla-Ascoli theorem, every bounded equicontinuous family has a uniformly convergent subsequence. Call the limit of this subsequence $u_+ : S^1 \to W$. This gives us the proposition.

■

Lemma 0.3.4 The limit u_+ is smooth, and a critical point of $\mathcal{A}_H$.

Proof. It actually suffices to prove that it is a critical point of $\mathcal{A}_H$, because satisfying the relation $\dot{u}_+ = X_t(u_+)$ is the same as being smooth, by Elliptic regularity, or some other result. So we show that u_+ satisfies

$$u_+(t) - u_+(0) - \int_0^t X_t(u_+(\tau))d\tau = 0$$

Notice here that we work in some ambient $\mathbb{R}^m$ in which W is embedded. Instead, we write this out as the limit, which gives

$$u_{s_k}(t) - u_{s_k}(0) - \int_0^t X_t(u_+(\tau))d\tau = \left(\int_0^t \dot{u}_{s_k} - X_t(u_{s_k}(\tau))d\tau\right) + \left(\int_0^t (X_t(u_{s_k}(\tau)) - X_t(u_+(\tau))d\tau\right).$$

The first term tends to zero from $\|\dot{u}_{s_k} - X_t(u_{s_k})\|_{L^2}$ tends to zero, and Cuachy Schwarz. The second term tends to zero from uniform continuity. ■

Lemma 0.3.5 The value $\mathcal{A}_H(u_{s_k})$ tends toward $\mathcal{A}_H(u_+)$.

Proof. The action is broken into two parts: the hamilton contribution of $\int H_t(u_{s_k})dt$ and the contribution from integrating over bounded disks. As u_{s_k} converges uniformly to u_+, we should expect the hamiltonian contribution to tend toward the desired value. For every k, choose extensions $\tilde{u}_{s_k} : D^2 \to W$ and $\tilde{u}_+ : D^2 \to W$. We need to show that

$$\int_{D^2} (\tilde{u}^*_{s_k} - \tilde{u}^*_+)\omega \to 0$$

Work in a small neighborhood of u_+ so that the form $\omega = d\lambda$ on this neighborhood. Pick k high enough so that there are homotopies from u_{s_k} to u_+ contained completely in this neighborhood. The difference $\int_{D^2}(\tilde{u}^*_{s_k} - \tilde{u}^*_+)\omega$ is given by the integral of ω over this homotopy cylinder, as the homotopy cylinder and two disks gives a sphere, and the integral of ω over every sphere is assumed to be 0. On this cylinder, the form ω is assumed to be exact, so we can instead integrate λ over each of the u_k using Stoke's theorem. By convergence of the loops in L^2, we have the desired result. ■

This proves the proposition.

Proposition 0.3.6 If every orbit is nondegenerate, there are only finitely many periodic orbits.

Proof. Periodic orbits are the intersection points of Δ with $\phi_1(\Delta)$ in $W \times W$. The nondegeneracy condition is the same as the transversality statement here. ■

Lemma 0.3.7 Fix $u \in \mathcal{M}$, and let s_k be any sequence tending to $+\infty$. Then there exists a subsequence $s_{k'}$ and a critical point u_+ in $\mathcal{A}_H$ such that $u_{s_{k'}} = u_+$.

Notice the difference between this and the earlier lemma, which found a sequence s_k from $\mathbb{R}$. Now, we are given a sequence (not necessarily convergent) and find a convergent subsequence.

Proof. We use the translation invariance of solutions. Define $u_{\cdot+s_k}$ to be the translated solutions. Due to the compactness of the space $\mathcal{M}$, we may take a subsequence that the sequence $u_{\cdot+s'_k}$ converges to some $v \in \mathcal{M}$. Fix some s_0. Then we have that

$$\mathcal{A}_H(v_{s_0}) = \lim_{k\to+\infty}(u_{s_0+s_k}) = \lim_{s\to\infty}\mathcal{A}_H(u_s)$$

the second equality comes from the proof that action behaves well with limits. This means that the action on v_{s_0} is independent of s_0, which tells us that v is a critical point of the action. Therefore we are done. ■

We are now in position to prove the whole theorem.
Because there are a finite number of critical points, we may take ϵ small so that neighborhoods of these critical points are disjoint in the loop space. For every solution u, we know that there is large enough s so that the ends of u past k must be contained in the union of these balls. Since the balls are disjoint, but the image of u is connected, we have that the tail of u is contained in these balls. Note that here we need that every sequence contains a convergent subsequence that goes to a ball– it is not enough to find a single subsequence that has this property as we do not know that u has something like uniform continuity.