1 Hochschild Cohomology and A_∞: Jeff Hicks

Here's the general strategy of what we would like to do.

- From the previous two talks, we have some hope of understanding the triangulated envelope of the Fukaya category by instead understanding the A_∞ relations on a set of generators. The plus side is that we need only understand the structure of a few Lagrangians. The downside is that understand even small products on arbitrary Lagrangians, we need to know the full A_∞ structure of the generators.
- There is a way to simplify the Fukaya category. Suppose $L_1, \dots, L_k$ is some set of Lagrangians in (X, ω), which bound a holomorphic polygon u contributing to some higher product in the Fukaya category of some surface. I can remove the count of this polygon (and in some ways simplify my Fukaya category) by putting a puncture in the surface exactly through where the polygon lived.
- More generally, given some divisor $D \subset X$, we may be able to compute $\mathrm{Fuk}(X \setminus D)$. Seidel's approach to proving Mirror symmetry outlined in [**seidel2002fukaya**] is to show that $\mathrm{Fuk}(X \setminus D)$ and $\mathrm{Fuk}(X)$ can be related by deformations of a certain kind.
- Summarizing:

  $$\text{Simple } A_\infty \text{ category } A \xrightarrow{\text{Extend by deformation}} \text{Complicated } A_\infty \text{ category}$$

  In good cases, we can show that the deformations of algebraic objects is classified by an object called the Hochschild homology. Namely, to a dg-category $\mathcal{A}$, we will associate a bigraded homology theory called the *Hochschild Group*

  $$HH^k(\mathcal{A})^j$$

  and show that $A_\infty\mathcal{S}(\mathcal{A})$, the set of A_∞ structures on $\mathcal{A}$ is determined (up to homotopy) by a deformation class.

1.1 Deformation of Algebras

My notes for this section were based on [**voronovlecture**].

Definition 1. *Let A be an algebra over k. A* formal deformation *of A is a $k[[t]]$ bilinear multiplication law:*

$$m_t : A[[t]] \otimes_{k[[t]]} A[[t]] \to A[[t]]$$

where $m^0(a,b)$ is the original multiplication on a, and m_t is associative. [1]

Generally, we will write the multiplication law as a power series:

$$m_t(a,b) = \sum_{k=0}^{\infty} t^k m_k(a,b).$$

Given an algebra A, we would like to know what kind of deformations it admits. One way to do this is to find deformations to kth degree, where we require associativity when we set $t^{k+1} = 0$.

Example 1. *What kind of deformations are there to first degree? Then our power series is truncated as:*

$$m_t(a,b) = m_0(a,b) + t m_1(a,b).$$

[1] This is not to be confused with the A_∞ multiplication index... yet.

Associativity of this equation is reduced to showing that

$$\begin{aligned}0 =& m_t(m_t(a,b),c) - m_t(a,m_t(b,c))\\ =& m_0(m_0(a,b)+tm_1(a,b),c)) + tm_1(m_0(a,b)+tm_1(a,b),c))\\ &- m_0(a,m_0(b,c)+tm_1(b,c)) - tm_1(a,m_0(b,c)+tm_1(b,c))\end{aligned}$$

As $t^2 = 0$ and m_0 is already associative

$$=t(m_0(m_1(a,b),c) + m_1(m_0(a,b),c) - m_0(a,m_1(b,c)) - m_1(a,m_0(b,c))$$

Satisfying this equation is enough to be a first order deformation.
One example of a first order deformation is given by derivations. Given any map $\phi : A \to A$, we can define the associated derivation as

$$m_\phi(a,b) = m_0(\phi(a),b)) - m_0(b,\phi(a)).$$

By the magic of plus and minus signs, $m_0 + tm_\phi$ is a first order deformation of A.

Notice that being a first order deformation does not in any way guarantee that you extend to an actual deformation of the algebra.

Question 1 (Motivating Question)**.** *When can we extend a first order deformation to a deformation?*

Since we have an object whose kernel is first order deformations, and the image of a different object which may be "boring" first order deformations, it is natural to set up a cohomology theory which classifies these.

Definition 2. *Let A be a k algebra. The* Hochschild complex $C^\bullet(A,A)$ *has*

- *As chain groups $C^k(A,A) := \hom(A^{\otimes n}, A)$.*
- *The differential is defined by*

$$\begin{aligned}(df)(a_0,\cdots,a_n) =& a_0 f(a_1,\ldots,a_n) +\\ &\sum_{i=0}(-1)^{i+1} f(a_0,\ldots,a_i a_{i+1},\ldots a_n)\\ &+ (-1)^{n+1} f(a_0,\ldots,a_{n-1})a_n.\end{aligned}$$

The cohomology of this theory is the *Hochschild cohomology.*

Claim 1. *$H^1(A,A)$ classifies derivations on A up to inner derivations (which are given by multiplication by an element.)*

Claim 2. *$H^2(A,A)$ classifies first order deformations of A up to derivations.*

The Hochschild cohomology is actually an algebra, equipped with the *Gerstenhaber bracket*

$$\begin{aligned}[-,-] : C^m(A,A) \otimes C^n(A,A) \to& C^{m+n-1}(A,A)\\ [f,g](a_0,\cdots,a_{m+n-1}) =& \sum_k \pm f(a_0,\ldots,a_k, g(a_{k+1},a_{k+n}),\ldots a_n)\\ &- \sum_j \pm g(a_0,\ldots,a_k, f(a_{k+1},a_{k+n}),\ldots a_n)\end{aligned}$$

where I have dropped signs. Importantly, one can check that if we force f and g to commute with t, then the associativity equations become

$$[m_t, m_t] = 0.$$

and we can write

$$df = [f, m_0]$$

where m_0 is the algebra multiplication. Expanding out $[m_t, m_t] = 0$, we get the following term by term expansion:

$$\begin{aligned}
[m_0, m_0] =&0\\
2dm_1 =&0\\
2dm_2 + [m_1, m_1] =&0
\end{aligned}$$

and so on. In particular, the obstruction to finding an m_2 extending the first order deformation m_1 is dependent on the exactness of $[m_1, m_1]$. This means that $H^3(A, A) = 0$, we can always extend to a second order deformation.
A send note is that our differential can be represented with the Gerstenhaber Bracket as

$$df = [m_0, f].$$

Theorem 1 (Hochschild-Extension)**.** *Suppose that $HH^k(A)$ vanishes for $k = 3$. Then $HH^2(A)$ parameterizes deformations of the algebra.*

Remark 1. *This should look suspiciously similar to language used to do things like construct deformations of complex structures by using the Kodaira Spencer map, etcetera.*

1.2 A_∞ Category

My notes for this section are based on [**abouzaid2013homological**]. Recall, an A_∞ category is a collection of objects $L_i \in Ob(\mathcal{A})$ and for each pair of objects, a graded space $\mathbb{K}$-module

$$\mathcal{A}(L_i, L_j)$$

along with k-multilinear composition maps

$$m^k \in \hom_{\mathbb{K}}^{2-k}(\bigotimes_{0=1}^{k-1} \mathcal{A}(L_i, L_{i+1}), \mathcal{A}(L_k, L_0)$$

satisfying the A_∞ relations:

$$\sum_{i+j+k=l} \pm m^l(\mathrm{id}^{\otimes i} \otimes m^j \otimes \mathrm{id}^{\otimes k}) = 0$$

Here, I have not specified the range of k. If $k \geq 1$, then we get a A_∞ category. If $k \geq 0$, we get a *curved* A_∞ category.
Let us reduce to the case when $\mathcal{A}$ is just a category. There are two questions that might interest us.

- When can we add in higher morphisms to $\mathcal{A}$ making it a A_∞ category.
- What are the deformations of the category structure on $\mathcal{A}$.

For a category[2], we can define a bigraded *Hochschild complex*

$$CC^{k+l}(\mathcal{A}^l) = \hom^l\left(\bigotimes_{0=1}^{k-1} \mathcal{A}(L_i, L_{i+1}), \mathcal{A}(L_k, L_0)\right)$$

[2]You can extend this to A_∞ categories, at the cost of more associativity terms, and more signs. For a full exposition, see [**seidel2008fukaya**]

Notice that A_∞ multiplication m^k is a $CC^2(\mathcal{A}^{2-k})$ cochain.
The differential on this cochain will be application of the composition law at every spot. Let $\phi \in CC^{k+l}(\mathcal{A}^1)$ be some cochain. Then

$$d\phi(a_1, \dots a_k) = m^2(a_1, \phi(a_2, \dots, a_k)$$
$$\pm \sum_{i=1}^{k-1} \phi(a_1, \dots m^2(a_i, a_{i+1}), \dots, a_k)$$
$$+ m^2(\phi(a_1, \dots, a_{k-1}), a_k)$$

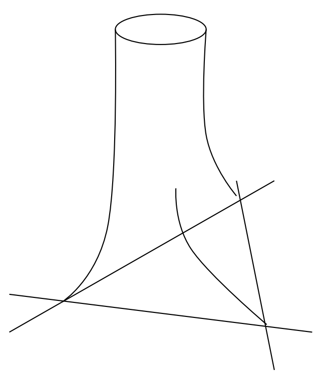
For the theory that we have set up here, there are similar deformation results as to those in the algebra case.

1.3 Geometric Interpretation of Hochschild cohomology

There is a map from symplectic cohomology to the Hochschild cohomology. In this section, we follow [**seidel2003homological**].
Let X be some symplectic manifold with nice proprieties, and pick D some divisor in X. Look at $X \smallsetminus U$, where U is some small neighborhood of D. We can give $X \smallsetminus U$ the structure of a symplectic manifold with contact-like boundary. In symplectic cohomology, we'll look at punctured Riemann surfaces with Reeb dynamics near the boundary. Roughly speaking, the generators are Reeb orbits, and the differential is given by counting holomorphic cylinders between those orbits.
So, how do we get a map from this to the Hochschild homology for the Fukaya category? Given an Reeb orbit o, and some set of intersections $\alpha \in \bigotimes_{i=0}^{k} \hom(L_i, L_{i+1})$, we define the map from Symplectic cohomology to the Hochschild complex by the count

$$\langle o, \alpha \rangle := \#\{\text{Punctured disks with boundary conditions}\}$$

of disks that look like this:



Seidel states that we should interpret these disks as counting the deformations we get by deforming the category geometrically along the divisor D.

1.4 Some Algebra and examples

Proposition 1. *Assume that $\mathcal{A}$ is a graded k-linear category, and*

$$HH^2(\mathcal{A}^j) = 0 \text{ for } j \leq -1 \text{ and } j \neq -l$$
$$HH^3(\mathcal{A}^j) = 0 \text{ for } j < -l.$$

Then the set of A^∞ structures (agreeing with m_1 and m_2)are exactly parameterized (up to homotopy) by deformations coming from $HH^2(A)^{-l}$.

Let's first define what a homotopy of A_∞ categories is:

Definition 3. *A* A_∞-functor *is a map* $\bar{f} : \mathcal{A} \to \mathcal{A}$ *between objects, and maps on the morphism spaces*

$$f_k : \otimes_{i=1}^{k-1} \mathcal{A}(X_i, X_{i+1}) \to \mathcal{A}(\bar{f}X_k, \bar{f}X_0)$$

satisfying the A_∞ *relations:*

$$\sum_r \sum_{i+j+k=l} \pm f_{l-j+1}(\mathrm{id}^{\otimes i} \otimes m^j \otimes \mathrm{id}^{\otimes k})$$

Two A_∞ *structures with multiplication* m *and* m' *are called* strictly homotopic *if there is another* A_∞ *functor, acting identically on objects, with* $f^1 = \mathrm{id}$.

Proof. Suppose that we would like to check that some product satisfies the A_∞ relations. We'll fix some $m^{l+2} \in HH^2(\mathcal{A}^{-l})$ which we would like to be our deformed multiplication. So, we need to fix in a whole A_∞ multiplication

$$m^2 + 0 + \cdots + m^{l+2} + m^{l+3} + \cdots$$

where we only know the first 2 non-zero terms. Let's suppose that we are trying to find m^k. The A_∞ constraint that m^k has to with relation to lower-order terms

$$dm^k = \text{An expression of } m^i \text{ for } 2 < i < k, \text{ equivalent to Gerstanharber Bracket}$$

This expression would be more complicated if we were not working in the nice case where $m^1 = 0$.

Claim 3. *This expression for this bracket is closed under the Hochschild differential.*

As the homology vanishes, we know we can always find m^k solving this equation. Inductively, we can build up the differential to solve this problem.
Now to prove the second claim of the proposition which is to show that all A_∞ structures on $\mathcal{A}$ are homotopic to one of these.
Let m' be an A_∞ structures. Let $m = m^2 + 0 + \cdots + (m')^{l+2} + m^{l+3} + \cdots$, where we've deformed the multiplication structure by m' at the first place where the Hochschild cohomology does not vanish. What would a strict homotopy have to satisfy? Since m' is suppose to be compatible with the category structure on $\mathcal{A}$, we know that $m^2 = (m')^2$, so the first order matching criteria is already filled. We are therefore looking for a collection of higher order functions satisfying the A_∞ functor relationships. This relationship can be written down as:

$$df^k = \text{Something like a Gerstanharber Bracket of } m \text{ and } m', \ f_i \text{ for } 1 < i < k$$

This bracket expression checks out to be Hochschild closed, and by the vanishing of cohomology, we can find solutions for df^k. □

Example 2 (The Sphere)**.** *Let's try to compute some Hochschild homology. Let's look at the sphere, and some Lagrangian on it. Since the symplectic form on the sphere is not exact, we'll need to constrain the set of Lagrangians we consider. In this scenario, we will work with the* balanced Lagrangians*, which are those which split the symplectic manifold into two parts with equal area. In this scenario, we have a well defined Fukaya category, where all of the objects are hamiltonian isotopic to the equatorial lagrangian. So we have one object,* L_1 *and the homology* $\mathcal{A}(L_1, L_1) = \{e, x\}$*, where* x *has degree* 1 *and* e *has degree* 0.
Let's take a look at $CC^{k+l}(\mathcal{A}^l)$*. This is maps of degree* l *from* k*-chains. Since* $\mathcal{A}(L_1, L_1)$ *only has degree* 0 *and* 1*, it means that the dimension of this space is* $\binom{k}{l} + \binom{k}{l+1}$.
Let's compute some Hochschild cohomology. A basis of $CC^{k+l}(\mathcal{A}^l)$ *is a string* e*'s and* x*'s of length* k*, with* l *or* $l+1$ x*'s depending on whether the string is mapped to* e *or* x.

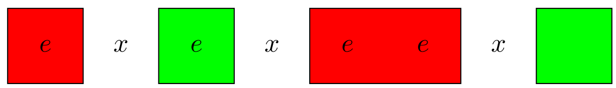
$$\underbrace{e \otimes x \otimes \cdots \otimes e}_{k \text{ things with } -l \ x's.} \quad \to e$$

$$\underbrace{e \otimes x \otimes \cdots \otimes e}_{k \text{ things with } -l+1 \ x's.} \quad \to x$$

Let's call the string η. Basically, the image of η is going to be determined by inserting e's in different places. Let's look at the image of such a string η. I'm going to forget about $\pm$ signs for a moment. The image of a string is the set of all strings where either

- *An e has been inserted in a place where the are an odd number of e's written consecutively in the string (and not at the start or end of the string.)*
- *An e has been inserted in at the start or end of the string, and the string η even number of e at the start or end consecutively.*

So, given a string η, mark all the places where you can insert e's.



Notice that the insertion of an e converts an area where you are allowed to insert an e to an area where you are not allowed to insert an e; the other regions stay the same. So (forgetting about signs and cancellation) on sees that the image of applying d to any such η produces the following successive images.

$$\begin{tikzcd}
& e \otimes x \otimes e \otimes x \otimes e \otimes e \otimes e \otimes x \arrow[dl] \arrow[dr] & \\
e \otimes x \otimes e \otimes x \otimes e \otimes e \otimes e \otimes e \otimes x \arrow[dr] & & e \otimes x \otimes e \otimes x \otimes e \otimes e \otimes e \otimes x \otimes e \arrow[dl] \\
& e \otimes x \otimes e \otimes x \otimes e \otimes e \otimes e \otimes e \otimes x &
\end{tikzcd}$$

You should get a cube with vertices indexed by the valid places to insert an e. At least with characteristic 2, this means that our differential squares to zero; I'm pretty sure that the $\pm$ signs work out in general shows that differential squares to zero.
This computation shows that the Hochschild cohomology of this category is zero, and therefore the only A_∞ structure that we can put on it (up to homotopy) is the trivial one.
However, we could expand our theory to allow for deformations by allowing μ^0 deformations. By our results, these should be unobstructed.
So the short story is once we define the μ_0 obstruction term, all the other terms are defined uniquely up to homotopy.
What does this μ^0 term represent? It gives us an idea of how big the upper and lower hemisphere are of the sphere. It also parameterizes the Lagrangian, as Lagrangians on the sphere are hamiltonian isotopic up to the difference in area between their two hemispheres.