Introduction

The goal of this talk is to exhibit some relations between 3 different approaches to topology: combinatorial / algebraic topology, topology detected by geometry / gauge theory, and topology that can be detected by symplectic / contact homology. The examples that we construct show the relationships between a series of knot invariants constructed with these three different approaches:

- On the geometry / gauge theoretic side, we study critical points of the Chern-Simons action for $SL(2,\mathbb{C})$. This corresponds to finding flat connections on $SL(2,\mathbb{C})$ principle bundles, which correspond to finding representations $SL(2,\mathbb{C})$ representations of the fundamental group. We can relate this to a classical geometry problem of studying hyperbolic knots. One viewpoint is that moduli space of critical points can be studied via the "representation variety," whose defining polynomial is the *A-polynomial* of a knot.

- On the combinatorial/ algebraic topology side, we study the *cord algebra*, which is an algebra based on the fundamental group of the knot. We can study the set of augmentations of this algebra, which gives us the *Augmentation variety* of the algebra. I'm not sure what the defining polynomial geometrically represents, but it is something that we might be interested in.

- On the symplectic / contact geometry side, we can study the Legendrian contact homology of the conormal legendrian to a knot. This invariant is pretty difficult to get our hands on, but we can simplify it by using the tools of augmentation, and get another example of an augmentation variety. These augmentations take on a physical interpretation via exact lagrangian fillings of the contact manifold. The set of augmentations can be classified by the Augmentation polynomial

As suggested by the summary, the A-polynomial, and Augmentation polynomials of the Cord algebra and Knot contact homology are all related. This points to two big themes:

- Both Gauge theory and Symplectic field theory give us ways of encoding topological data in algebraic forms.

- Partial validation of the physics - motivated relations between field theory and string theory.

This talk is broken into 3 parts: Symplectic, Gauge and Topological (in that order.) I think the symplectic geometry requires the most machinery to set up, so we'll cover that on the first day. On the second day, we'll skim through the Gauge theory and topological constructions, and tie together all three stories.

1 The Symplectic Story

1.1 Introduction and Notation

Today, we're going to look at a knot contact homology, a knot homology theory based on Legendrian contact homology.

- Given $K \subset \mathbb{R}^3$, we can construct Λ, a Legendrian submanifold inside of $S^*\mathbb{R}^3$, the unit sphere in the cotangent bundle of $T^*\mathbb{R}^3$. The Legendrian isotopy class of this submanifold is a knot invariant. However, it is unknown if the Legendrian isotopy class of Λ is a complete invariant of the knot.
- In this scenario, we have a differential graded algebra associated to Λ. This is a weaker invariant than Λ itself, but it is at least a bit computable.
- We will then pull some homological invariants from this DGA, and relate them back to the geometry of the Legendrian.

First, let's set up some notation and recall some basic definitions.

Definition 1. *A contact manifold is a pair (M, α) where*

- *M is a 2k+1 manifold*
- *α is a 1 form satisfying $\alpha \wedge (d\alpha)^k \neq 0$.*

Given a fixed contact form, a Reeb vector field *R_λ is the unique vector field satisfying*

$$d\alpha(R_\alpha, -) = 0$$
$$\alpha(R_\alpha) = 1$$

There are 2 basic examples of Contact manifolds that we will look at.

Example 1. *Let N be a k dimensional manifold. The* first Jet bundle of real valued functions *on N is the bundle whose fiber at each point is the set of all functions to $\mathbb{R}$ under the equivalence relation of "having same first order Taylor expansion." We denote this bundle as J^1N.*
Notice that J^1N is isomorphic to $T^*N \times \mathbb{R}$, where the first coordinate records the derivatives of the function, and the second coordinate records the value of the function. We define the contact structure at (λ, t) to be

$$\alpha = \lambda + dt.$$

Example 2. *Let N be a k dimensional manifold, and fix a Riemannian metric on N. Let S^*N be the* unit cotangent bundle*, consisting of all differential one forms with norm 1. Then we can define the canonical contact form at a point (x, λ) by*

$$\alpha = \lambda$$

which is a contact form. Notice here that the Reeb vector flow R_α generates the geodesic flow of the metric on the unit cotangent bundle.

We will primarily work with the first example, where Legendrian contact homology is well defined.

Definition 2. *Let (M, α) be a contact manifold. A* Legendrian *$\Lambda \subset M$ is a k-dimensional submanifold such that $\alpha|_\Lambda = 0$.*

We should think of these as the natural counterparts to Lagrangian submanifolds of a symplectic submanifold. In fact, there is nice way to take the contact geometry story and upgrade it to a symplectic one.

Definition 3. *Let (M, α) be a contact submanifold. The* symplectization *of M is the symplectic manifold $(M \times \mathbb{R}, \omega)$, where $\omega = d(e^t\lambda)$. To every Legendrian Λ of M, we get a Lagrangian by taking $\Lambda \times \mathbb{R}$.*

This leads us to the idea of Legendrian Contact Homology: to study the $\Lambda \subset M$ a Legendrian submanifold (or maybe its Legendrian isotopy class,) we might be able to study invariants of $\Lambda \times \mathbb{R}$ (like Lagrangian Floer Theory.) While the technical framework of this theory is fairly difficult, our goal today will be to understand the construction and some applications of this tool. Notice that the symplectization of a contact manifold is not compact.

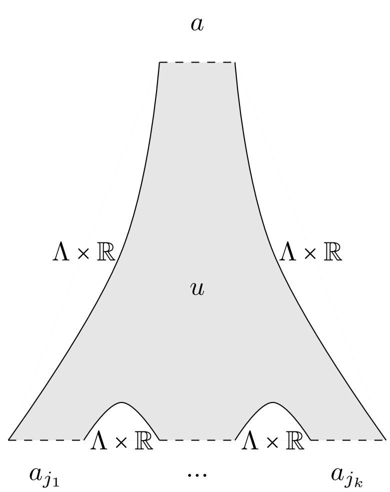
1.2 Legendrian Contact Homology

We already have the basics set up to study Legendrian contact homology. At this point, the machinery of Legendrian contact homology is only well defined for Jet bundles. The set up of our theory take as an input a Legendrian Λ, and outputs a differential graded algebra. [1]

Definition 4. *Let (M, α) be a Jet bundle, and Λ a Legendrian submanifold of M.*

1. *The* Algebra *associated to the pair (M, Λ) will be defined over the ring $R = \mathbb{Z}[H_2(V, \Lambda)]$, which is the group ring of the relative homology.*
   The algebra associated to Λ will encode some of the contact geometry of Λ. Let $a_1, \dots a_k$ be the set of self- Reeb chords of Λ. Then
   $$\mathcal{A} := R\langle a_1, \dots a_k \rangle$$
   the free non-commutative algebra generated by the Reeb chords. We can turn this into a graded algebra *by giving each of the generators a_i an index; for us, $|a_i|$ will be taken to be the Conley Zehnder index minus 1. The index of an element in the ring R is defined to be 0.*

2. *The* differential *of this algebra will be given by a count of holomorphic strips in the symplectization. For each $(a; a_J) := (a, a_{j_1}, \dots, a_{j_k})$ and $\beta \in H_2(V, \Lambda)$ we define the space $\mathcal{M}_\beta(a; a_J)$ to be the count of strips in this configuration*



where the class of u in the relative homology is equal to β. The class β is given by projection from $H_2(M \times \mathbb{R}, \Lambda \times \mathbb{R}) \to H_2(M, \Lambda)$, where we cap off each Reeb chord with a preassigned half disk. We then define

$$\partial(a_i) := \sum_{\beta, J} |\mathcal{M}_\beta(a; a_J)| t^\beta a_J.$$

A summary of results of this DGA:

[1] Recall, a *differential graded algebra* is an algebra $\mathcal{A}$, and a map $\partial : \mathcal{A} \to \mathcal{A}$, such that $\partial^2 = 0$ and $\partial(ab) = a\partial b + b\partial a$.

Theorem 1. *$(\mathcal{A}, \partial)$ is a DGA, whose stable-tame isomorphism type is independent of contact structure, contact form and J, and Legendrian isotopy. In particular,*

$$HC_*(M, \Lambda) := H_*(\mathcal{A}, \partial)$$

is an invariant of Λ up to Legendrian isotopy.

In practice, this is incredibly difficult to calculate. Notice that while this is a *DGA*, it is a *non-commutative DGA*. It also has a *non-linear differential* in the sense that the differential has a Leibniz like behavior over the product.

Example 3. *Here is an example that we will work with for the rest of the talk. Let's look at the contact manifold $M = J^1 S^2$. Since this is a trivial fibration, we have that $J^1 S^2 = S^* \mathbb{R}^3$. Let K be a knot in $\mathbb{R}^3$. The unit conormal bundle to our knot $N^* K$, is a Legendrian of $S^* \mathbb{R}^3$. We therefore get for every knot K, a legendrian submanifold*

$$\Lambda_K \subset J^1 S^2$$

for which we can now study $HC_*(\Lambda_K, M)$. This is an invariant of the Legendrian isotopy class of the conormal bundle.

1.3 Augmentations

One way to simplify this theory is to supply an *augmentation* to this differential graded Algebra.

Definition 5. *An* augmentation to a ring *S of a DGA (over a ring R) $(\mathcal{A}, \partial)$ is a graded unital ring homomorphism*

$$\epsilon : \mathcal{A} \to S$$

so that $\epsilon \circ \partial = 0$.

The main difficulty of working with Legendrian contact homology is that the differential is non-linear. A choice of augmentation allows you to construct a linearized version of the homology to work with instead.

Definition 6. *Let $\epsilon : \mathcal{A} \to S$ be a augmentation. Suppose that $\mathcal{A} = R\langle a_1, \dots, a_k \rangle$. Then we extend ϵ to a map*

$$\epsilon_S : S\langle a_1, \dots a_n \rangle \to S$$

Then $C_*^{lin} := (\ker \epsilon s) / \ker(\epsilon s^2)$ is a finite graded module (generated by the a_i with degree not equal to 0) and is graded.
The map ∂ descends to a map on C_*^{lin}. Notice that now ∂ is linear, and it gives us the linearized homology *$H_*^{lin}(M, \Lambda, \epsilon)$*

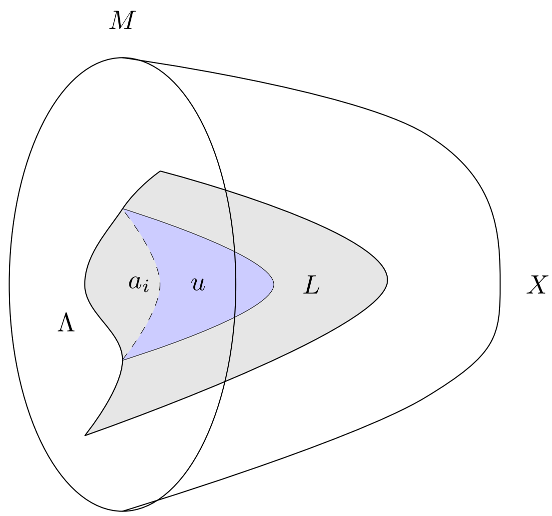
It is not the case that a linearization always exists! Here is maybe an intuition for what a linearization does:

- Take the map $\partial : \mathcal{A} \to \mathcal{A}$, and split it as a sum $\partial = \sum_{l=0}^{\infty} \partial_l$, where l is word length of $\partial(a)$ in the generators of $\mathcal{A}$.
- In particular if ∂_0, which maps $\mathcal{A} \to R$ is zero, then we have that $\partial_1^2 = 0$, so the restriction to the *module*

$$\partial_1 : R\{a_1, \dots a_k\} \to R\{a_1, \dots a_k\}$$

  is now a chain complex over R. This would be very good.
- Unfortunately this is rarely the case. You can hope to fix this by adding in a *augmentation*. If you are familiar with the language of A_∞ algebras, you should think of this as finding μ_0 term that turns our *DGA* into an actual chain complex when the A_∞ algebra is weakly unobstructed. This can give us some intuition for a geometric origin for constructing augmentations.

Based on the previous discussion, we have the following geometric intuition for constructing an augmentation for a Legendrian contact homology. Let X be a symplectic filling for M, that is, a symplectic manifold with contact boundary given by M. Furthermore, let's pick a Lagrangian L inside X so that $\partial L = \Lambda$.
To each a_i, we can now associate a class in $H_2(X, L)$, given by the count of holomorphic disks u with boundary in L, and one puncture limiting to a_i.



This defines a map from $\mathcal{A} \to Z[H_2(X, L)]$ which on generators gives this count, and on coefficients is the usual map between $H_2(M, \Lambda) \to H_2(X, L)$. One can check that this is an augmentation in a similar way to checking that $\partial^2 = 0$.

Remark 1. *By picking different augmentations of Legendrian contact homology, we can construct invariants of Legendrians. There is another interesting invariant that we could study– which is what kinds of Augmentations can we attach to the Legendrian DGA. This is how we will construct our invariants.*

1.4 Related Invariants

There are two related invariants that we might want to consider:

Example 4. *The* A-polynomial *records information related to the representations of the knot group. Specifically, if $\phi : \pi(S^3 \setminus K) \to SL_2(\mathbb{C})$ is a representation, you can associate to the meridional and longitudinal loops complex eigenvalues of their corresponding matrices: λ, μ. These eigenvalues satisfy an algebraic relation, and permissible values of λ, μ form a variety in $(\mathbb{C}^*)^2$. Taking a defining polynomial for this variety gives you the A-polynomial, $A_k(\lambda, \mu)$. It turns out that the A-polynomial always divides the 2-dimensional augmentation polynomial.*
Notice that the A-polynomial is telling you something about $SL_2(\mathbb{C})$ principle bundles equipped with flat connections.

Example 5. *Given a knot K, you can consider the set of all cords from the knot to itself, up to homotopy. There is a algebra generated on these classes of cords, called the Cord Algebra of K. The Cord algebra of K can be computed using the knot contact homology explicitly.*

2 Gauge Theory and Knot Invariants

Today, we're going to take a look at the A-polynomial of a knot. Our general outline is the following:

- Start with some classical topology problem that we are interested in.
- State the problem symplectically, where symplectic geometry counts the topological relations that we are interested in.

2.1 Some History: Hyperbolic Knots

The classical problem was interested in hyperbolic knots. Recall, a knot is called *hyperbolic* if it's complement in S^3 admits a metric with constant negative curvature.
The *holonomy group* of a hyperbolic knot is defined to be the group of all parallel transports from a point back to itself along a contractible loop.

$$Hol_p^0 := \{P_\gamma \mid \gamma \text{ is a contractible loop from } p \text{ to itself}\}$$

This is a normal subgroup of the full holonomy group, and so we have a quotient

$$Mon(S^3 \smallsetminus K) = Hol_p / Hol_p^0$$

and a *monodromy representation* from

$$\phi : \pi_1(S^3 \smallsetminus K) \to Mon(S^3 \smallsetminus K)$$

Now, the monodromy group can be associated with a hyperbolic isometry (this is weird!) so we have a map

$$\phi : \pi_1(S^3 \smallsetminus K) \to Mon(S^3 \smallsetminus K) = Isom(\mathbb{H}^3) = PSL(2, \mathbb{C})$$

One way to see this map is that $\mathbb{H}^3$ is the universal cover of $S^3 \smallsetminus K$, so $S^3 \smallsetminus K = \mathbb{H}^3/\Gamma$, where Γ is monodromy group. In this particular case, we can take a lift of this map to a map to SL(2,$\mathbb{C}$). So our process takes a hyperbolic knot, and produces a flat $SL(2, \mathbb{C})$ bundle.
While not every knot is hyperbolic, we ask questions in general about flat $SL(2, \mathbb{C})$ connections, which exist for every knot. if you are a physicist, you may be interested in this quantity because it connects to Chern-Simons theory. Recall that the *Chern Simons form* of a $sl(2, \mathbb{C})$-valued 1-form A is given by

$$CS_A = Tr(F \wedge A + 2/3 A \wedge A \wedge A)$$

and that the action associated to a form A is

$$S(A) := \int_M CS_A$$

The critical points of this action are the flat connections. For this reason, I'm going to say that the invariants coming from this section have a gauge-theory "flavor"

2.2 A-polynomial

Maybe another reason to think about $SL(2, \mathbb{C})$ representations of knot group is because they are much easier to get our hands on than the knot group itself. Every knot has a distinguished subgroup, the *periphery subgroup*, generated by the longitudinal and meridional directions of the knot. It is an abelian $\mathbb{Z} \times \mathbb{Z}$ sitting inside of $\pi_1(S^3 \smallsetminus K)$.
Therefore, every representation of the knot group into $SL(2, \mathbb{C})$ gives us a representation of this periphery group, and the representations of $\rho : \mathbb{Z} \times \mathbb{Z} \to SL(2, \mathbb{C})$ have some nice invariants rising out of them. Picking generators (μ, λ) of the periphery subgroup, we can look at the eigenvalues associated to $\rho(\mu)$ and $\rho(\lambda)$. Let's call these eigenvalues L and M.

Claim 1. *L and M have to satisfy some algebraic relations. The set of all such L and M for an algebraic variety. The defining polynomial of the Zariski closure of this variety is called the* A -polynomial*, which we denote*

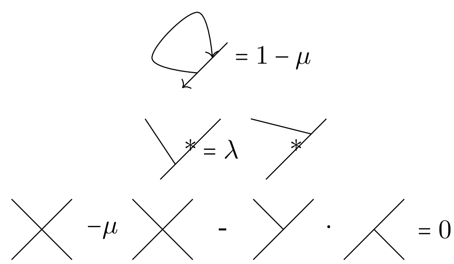
$$A_K(L, M).$$

I don't know a lot about this invariant, but it has some remarkable properties in hyperbolic geometry. Perhaps someone who knows more about hyperbolic geometry can talk about this, but for instance, if you take the polynomial, and plot Newton Polygon of the polynomial (lower convex hull of the non-zero coefficients of the polynomial), the slopes of this shape have same slopes as the *incompressible surfaces* of the knot complement. An incompressible surface is an map $\Sigma \to S^3 \smallsetminus K$, which is injective on fundamental groups. The slopes of the boundary curve in the homology of the boundary surface (with basis m and l) are the boundary slopes.
So, clearly, there is a lot of data contained in this polynomial.

3 Classical Topology

3.1 Cord Algebra

There is a slightly more classical topological viewpoint that one might take to understand the A-polynomial. Given a knot K with a marked point $*$, a *cord* of K is a path with endpoints on $K \smallsetminus *$ and interior in $S^3 \smallsetminus K$. Let $\mathcal{A}_K$ be the free algebra over $R_0 = \mathbb{Z}(\lambda, \mu)$ generated by homotopy classes of cords. We define the *cord algebra* to be the algebra $\mathcal{A}_K/I$, where I is the ideal generated by the following relations:



Notice that these μ and λ are the same as the meridional and longitudinal generators of the periphery subgroup. In fact, a representation of the knot group into $SL(2, \mathbb{C})$ extends to an augmentation of the cord algebra, which is a homomorphism

$$\epsilon : \mathcal{A}_K \to \mathbb{C}.$$

Claim 2. *Suppose that $\rho : \pi_1(S^3 \smallsetminus \mathcal{K}) \to SL(2, \mathbb{C})$ is a representation. Then we have a homomorphism by sending λ and μ to the Values of M and L. You send a class of chord γ to $(1 - \mu_0)[\rho[\gamma]_c$, where $\rho[\gamma]_c$ is the upper left entry in the representation of γ.*

Proof. We sketch an outline of a proof here:

- Take the set of cords and identify them with $\pi_1(S^3 \smallsetminus K)$, by gluing the ends of a chord by a push off of the knot.
- We then see that $\mathcal{A}_k$ is isomorphic to $R_0^{\otimes \pi_1(S^3 \smallsetminus K)}/I$, where I is generated by the relations

$$\begin{aligned} [e] =& 1 - \mu \\ [\gamma l] =& [l\gamma] = \lambda[\gamma] \\ [\gamma m] =& [m\gamma] = \mu[\gamma] \\ [\gamma\gamma'] - [\gamma m \gamma'] - [\gamma][\gamma'] =& 0 \end{aligned}$$

In order to show that we have an algebra homomorphism, we need to show that these relations are preserved by the augmentation.
The first three relations are clear from construction. For the fourth relation, notice that

$$
\begin{aligned}
\epsilon[\gamma\gamma'] &= (1-\mu_0)(\gamma_{11}\gamma'_{11} + \gamma_{12}\gamma'_{21}) \\
\epsilon[\gamma m \gamma'] &= (1-\mu_0)(\mu_0\gamma_{11}\gamma'_{11} + \gamma_{11}\gamma'_{21} + m^{-1}\gamma_{12}\gamma'_{21}) \\
\epsilon[\gamma][\gamma'] &= (1-\mu_0)^2\gamma_{11}\gamma'_{11}
\end{aligned}
$$

Where are three terms that cancel out. So, this gives us a Augmentation. ☐

4 Tying it all together

Recall that we had looked at Augmentations before in the context of Knot Contact homology, which was a homology theory whose

- Generators were self Reeb chords of the unit conormal bundle of a knot
- Differentials counted holomorphic disks with punctures in the symplectization.

This gave us a differential graded algebra $(\mathcal{A}, \partial)$, which was difficult to study. To make aid our understanding of this differential graded algebra, we introduced a homological algebra tool, an Augmentation, which was a graded ring homomorphism $\epsilon : \mathcal{A} \to \mathbb{C}$, where $\epsilon \circ \partial = 0$. I want to show that this DGA is related to the cord algebra we've been discussing.

Claim 3. $H^0(\mathcal{A}, \partial)|_{U=1} = \mathcal{A}_K$

Proof. We outline a proof here. Recall that the cord algebra is generated by cords.

Claim 4. *The cord algebra can, in fact, be generated by binormal geodesic cords(which are Reeb self chords)*

So, the generators of $\mathcal{A}_K$ can be expressed by the generators of the knot DGA. In fact, the first and 3 relations of the cord algebra give us these relations.
However, the cord algebra has many more relations given by the homotopy of cords. Using the machinery of gradient flow trees, one can show that these homotopies can be realized with j-holomorphic disks being counted in the Lagrangian Contact Homology. ☐

This gives us an interesting result:

Claim 5. *Augmentations of the cord algebra extend to augmentations of the knot DGA with $U = 1$.*

In the case of a the knot Λ_K inside of $S^*\mathbb{R}^3$, there is a polynomial that captures the Augmentations to $\mathbb{C}$ of the knot.
Let's suppose that we would like to construct an augmentation of $(\mathcal{A}_K, \partial)$, the DGA associated to the Legendrian contact homology of $(S^*\mathbb{R}^3, \Lambda)$.
The coefficient ring for this DGA is given by

$$\mathbb{Z}[H_2(S^*\mathbb{R}^3, \Lambda_K)]$$

The relative homology has 3 generators, which can be seen from the fact that Λ_K is a torus. We will call these generators λ, μ, U where

- λ corresponds to the longitudinal direction of the knot
- μ corresponds to the meridional direction along the torus.
- U is the generator of S^2.

Now, the coefficient ring is $\mathbb{Z}(\mu, \lambda, U)$. Any augmentation the knot to $\mathbb{C}$ will in particular have to be a map

$$\epsilon_{\mathbb{C}} : \mathbb{Z}[H_2(S^*\mathbb{R}^3, \Lambda_K)] = \mathbb{Z}(\mu, \lambda, U \to C$$

which is determined by the values $\epsilon(\mu), \epsilon(\lambda), \epsilon(U)$.

Claim 6. *For given K, the set of such $\epsilon(\mu), \epsilon(\lambda), \epsilon(U)$ is a variety in $(\mathbb{C}^*)^3$.*

It is conjectured that this variety can be captured with a polynomial. If we take the closure of this variety, and its maximal-dimension part is codimension 1, then we can describe this variety as the zero locus of a polynomial.

Definition 7. *Whenever this is true, let's call this polynomial $Aug_K(\lambda, \mu, U)$.*

Conjecture 1. *This polynomial is always well defined*

How should we think of this polynomial geometrically? Recall that every time we had a Lagrangian filling to the Legendrian, we got an augmentation by taking a count of disks. This means that those Lagrangian fillings must satisfy some conditions given by the augmentation polynomial.
We can simplify this augmentation polynomial to the *2 variable augmentation polynomial* by considering the slice of Augmentation variety which is constrained by $\epsilon(U) = 1$. From here, we get a polynomial $Aug_k(\lambda, \mu)$, which is conjectured to be the same as when $U = 1$.

A hope is that the augmentation polynomial is telling us about lagrangian fillings corresponding to the symplectic manifold $\mathbb{R}^4 \times S^2$.
However, there are different ways that you may hope to add lagrangian fillings. One possibility is thinking of this space as $S^*\mathbb{R}^3$, which means that this can be filled with the $D^3 \times \mathbb{R}^3$. The two dimensional Augmentation polynomial should tell us something about Lagrangian fillings in this setting instead.