1 Outline

1.1 Historical Motivation for ECH

ECH was motivated to fill in the symplectic side of the following picture:

Here is a rough outline of the ideas between these two theories:

Taube's Gromov Invariant	SW Invariants
- Defined for symplectic 4-manifold M.	- Defined for 4- manifold
- Inputs a class in homology $\Gamma \in H_2(M)$	- Inputs a Spin_c $\mathfrak{s}$ structure on M
- Pick J an almost complex structure and perturbations	-Pick g a metric and perturbations
- Count J holomorphic currents representing Γ	- Count solutions to SW equations over $\mathfrak{s}$
-Is independent of choice of J	-Is independent of choice of g

These two invariants completely determine each other. Notice that there is a key difference between Taube's Gromov Invariant and the Gromov-Witten invariants, in that one counts pseudo-holomorphic curves, and the other counts pseudo-holomorphic currents. This small difference leads to all of the difficulties that we are trying to solve using polyfolds.
Kronheimer and Mwroka constructed Seiberg-Witten Floer homology, which is a Floer theory coming from the Seiberg-Witten functional on 3-manifolds. Embedded Contact homology was constructed to provide a symplectic-type invariant which is isomorphic to this theory.

EC Homology	SWF Homology
- Defined for contact 3-manifold M.	- Defined for 3- manifold
- Inputs a class in homology $\Gamma \in H_1(M)$	- Inputs a Spin_c $\mathfrak{s}$ structure on M
- Pick J an almost complex structure and perturbations	-Pick g a metric and perturbations
- Count J holomorphic currents representing A	- Count solutions to SW equations over $\mathfrak{s}$
- Differential from Gromov invariant for $M \times R$	- Differential from SW invariant for $M \times \mathbb{R}$
- ________________________________	-Cobordisms define maps by SW invariant for X
- ________________________________	-Is independent of choice of g

Let's quickly review ECH the construction.

1.2 Outline of ECH Construction

Let M be a 3-manifold. In order to define the embedded contact homology, on M, we need 3 additional pieces of data:

1. A contact λ structure on M.
2. A class Γ in the homology $H_1(M)$
3. An almost complex structure J on $M \times \mathbb{R}$ which is compatible with both the contact structure and the product manifold topology.
4. (Optional) L, an energy bound on the action of Reeb orbits, which in many cases makes computation of ECH easier, by allowing certain kinds of perturbations.

Recall that the chain complex for ECH is generated by orbit sets $\alpha = \{(m_i, \alpha_i)\}_{i \in I}$, where α_i are nondegenerate Reeb orbits of the contact structure, $m_i \geq 1$, satisfying

$$\Gamma = \sum_{i \in I} m_i [A_i]$$

and $m_i = 1$ whenever $[A_i]$ is hyperbolic. We define $ECC_*^L(M, \lambda, \Gamma, J)$ to be the $\mathbb{Z}/2$ module freely generated on the orbit sets α, graded by the ECH index.
The philosophy for constructing the differential and showing independence of contact structure is the following:

(a) The differential should count holomorphic currents in $M \times \mathbb{R}$ which asymptotically approach orbit sets in M^+ and M^-. These currents should be counted up to the action of $\mathbb{R}$ translation.

(b) We show that the differential is well defined as it is the signed count of boundary points of holomorphic currents of a 1-dimensional space.

(c) A cobordism $W : M_+ \to M_-$ should count holomorphic currents in W which asymptotically approach orbit sets in M^+ and M^-.

Making these counts work is difficult in principal because we have hard time counting holomorphic currents. In place of holomorphic currents, we work with holomorphic curves that represent these currents. How does this translation work into the counting arguments?

(a) When counting holomorphic currents in $M \times \mathbb{R}$ with index 1, all holomorphic currents can be described by simple holomorphic curves. This means we can largely avoid the problem of multiply covered holomorphic currents.

(b) When trying to show that $d^2 = 0$, we can show that holomorphic curves will converge to broken holomorphic curves with 1 multiply covered component. Through hard arguments, we are able to control this case.

(c) Index 0 holomorphic currents in a cobordism can be of any shape or form. We do not know how to properly count these yet.

1.3 Goal of this Talk

The goal of this talk is to highlight the differences between the differential construction and the cobordism map construction, and to give examples of why the cobordism construction fails. Finally, we would like to outline the general process for constructing the cobordism map from SWF theory.

2 Proof that ∂ is well defined

In order to appreciate the difficulties of constructing the cobordism map, we first show how we sidestep these difficulties in the case of constructing a the map on a cylinder.

Fact 2.1
Properties of Currents with Low ECH Index

The following is true on a generic set of almost complex structures. Given α and β two orbit sets, and $\mathcal{C} \in \mathcal{M}(\alpha, \beta)$ a J-holomorphic current in $\mathbb{R} \times Y$ we have:

(a) The ECH index $I(\mathcal{C}) \geq 0$, with equality if and only if $\mathcal{C}$ is union of cylinders with multiplicities.

(b) If $I(\mathcal{C}) = 1$, then $\mathcal{C}$ is a union of trivial cylinders, and a current $\mathcal{C}_1$ of Fredholm and ECH index 1. Furthermore, this curve must be embedded.

Sketch of Proof. The trick here is to notice that in the case of $Y \times \mathbb{R}$, you can take the components of a holomorphic current and R translate them so that they are not multiple covers (unless the current is a cylinder.) This means that holomorphic currents can be assumed to be somewhere injective. When a holomorphic current is somewhere injective, we have

$$\operatorname{ind}(C) \leq I(C) - 2\delta(C)$$

In the case of a cylinder, we can reduce the multiply covered case to the singly covered case. We know that the ECH index is only dependent on the relative homology class of a curve. We first work in the case where C has no multiply covered cylinders. Suppose that C is some multiply covered current. By taking $\mathbb{R}$ translations of its components, we get a curve which is not multiply covered (as C contains no cylinders), but still represents the same homology class. We will compute the ECH index of this curve instead.
Since this curve is somewhere injective, we can use the index inequality. For injective curves, ECH index 0 implies cylindrically, because $\mathbb{R}$ translations of the curve would give us Fredholm index 1, which must be less than the ECH index. This gives us the first claim.
For the second claim, suppose that $\mathcal{C}$ has multiple components. If it has ECH index 1, then it can't be a union of cylinders. But it cannot have more than one component with ECH index 1, because the ECH index is additive. This means that it contains exactly one component of ECH index 1. Furthermore, $\mathbb{R}$ translations of this non-cylindrical curve will give us Fredholm index at least 1, and so $\delta(C) = 0$. ∎

This key observation allows us to avoid most of this multiply covered business in the case of low index. Since the differential in ECH is the ECH-index 1 count of curves, we know that our arguments for defining the differential can avoid references to multiply covered components.
We get additional structure on the shape of holomorphic curves by a compactness result of Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder.

Theorem 2.2 BEHWZ

The space of stable holomorphic buildings (broken holomorphic curves) with genus g and μ connected components of bounded energy is compact.

We would like to apply the compactness result of Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder, however their compactness result applies to holomorphic buildings of known genus. It is not too much work to upgrade this result though:

Lemma 2.3

Any sequence $\mathcal{C}_\nu$ of holomorphic currents representing a sequence in $\mathcal{M}(\alpha, \beta)/\mathbb{R}$ converges to a broken holomorphic current.

Proof. To translate BEHWZ into this case, we restrict to intervals of $Y \times \mathbb{R}$, and then let those intervals grow. Let $\mathcal{C}_\nu$ be a sequence of holomorphic currents representing some sequence. Then by restricting to an interval $[a, b]$, we can use the result of BEHWZ to get a holomorphic current $\mathcal{C}_{[a.b]}$ which a subsequence $\mathcal{C}_\nu$ converge to. We can get this for every $[a, b]$, and by diagonalize we can get convergence to a holomorphic current on all of $[a, b]$. Let's call this current $\hat{\mathcal{C}}$.
What we do not know that $\hat{\mathcal{C}}$ is a current between α and β. We do know that it goes between curves α', β' with energy bounds

$$E(\alpha) \geq E(\alpha') \geq E(\beta') \geq E(\beta).$$

which can be seen by taking translations of $\hat{\mathcal{C}}$.
By appropriately picking our translations to have action close to that of α in the slice $Y \times \{0\}$, we can "pick off" some non trivial component of the limiting curve. Repeating this over and over allows us to read off the different components of the limiting curve, and by action considerations we can only do this finitely many times.

∎

Lemma 2.4

Suppose that $\mathcal{C}_\nu$ converges to a broken holomorphic current $\mathcal{C}^i$. Then

$$\sum_i [\mathcal{C}^i] = [\mathcal{C}_\nu]$$

Proof. Suppose not. Then take another subsequence in the above argument to get another level. ∎

Lemma 2.5 ∂ is well defined

If α and β are orbit sets, the $\mathcal{M}_1(\alpha, \beta)/\mathbb{R}$ is finite.

Sketch of Proof. Suppose there is an infinite number of distinct elements in $\mathcal{M}_1(\alpha, \beta)/\mathbb{R}$. By the key observation, we can assume that the difference is in an injective component with index 1. Taking a convergent subsequence, we may assume that all of these represent the same homology class. Therefore, we may take another subsequence that consists of curves of the same genus. Therefore we may apply BEHWZ to get another subsequence that converges in the sense of holomorphic curves to a broken holomorphic curve. Let this broken holomorphic curve be $(u^0, \ldots, u^k)$.
By the additivity of ECH index, we know that $\sum_i I(u^i) = 1$. By the key observation, we have that one of these curves has ECH index 1, and the rest have ECH index 0.
By additivity of Fredholm index, we have that $\sum_i \operatorname{ind}(u^i) = 1$. Take a curve with Fredholm index zero. By carefully keeping track of partition numbers, one sees that these curves are just the identity. Therefore there is only one curve. This curve has ECH index 1, Fredholm index 1 and suppose to be an isolated point in the moduli space. But this is suppose to be the limit of curves, a contradiction. ∎

3 Why Cobordisms are Hard

Let's first describe the thing that we would like to do.

Definition 3.1 An *exact symplectic cobordism* from (Y_+, λ_+) to (Y_-, λ_-) is a compact symplectic four manifold (X, ω) with boundary

$$\partial = Y_+ \sqcup Y_-$$

such that $\omega|_{Y_\pm} = d\lambda_\pm$.

Orientations here are important, as holomorphic curves in the exact symplectic cobordism decrease action from the side to the − side. For these reason, we cal the + and − side the "convex" and "concave" boundaries of X. Given such a cobordism, there is a unique vector field ρ such that $i_\rho \omega = \lambda$ which points inward along the boundaries. Then there is a small neighborhood of the boundary for which $\omega = de^s\lambda$, and s points in the ρ direction. We can attach semi-infinite cylindrical ends to the boundary with symplectic form $de^s\lambda$. We call this the symmetrization completion of X, and denote it $\bar{X}$.
Giving $\bar{X}$ an admissible almost complex structure J, we would like to bound J-holomorphic curves in $\bar{X}$ with positive ends at Y_+ and negative ends at Y_-. Given a curve C, the dimension of the moduli space associated to the curve is given by an index formula

$$\operatorname{ind}(C) = -\chi(C) + 2c_\tau(C) + \sum_{i=1}^{k} CZ_\tau(\gamma_i^+) - \sum_{i=1}^{l} CZ_\tau(\gamma_i^-)$$

where the relative first Chern class is defined by

$$c_\tau(C)c_1(\det(TX)|_C, \tau) \in \mathbb{Z}$$

where τ is a trivialization of the contact structure over the Reeb orbits. Similarly, we can define an ECH index for currents in $\bar{X}$ representing class $Z \in H_2(\bar{X}, \alpha, \beta)$

$$I(\alpha, \beta, Z) = c_\tau(Z) + Q_\tau(Z) + CZ_\tau^I(\alpha\beta)$$

We would like to define the differential between two orbit sets α and β to be the count of ECH index 0 J holomorphic curves between α and β. There are two problems with this:

- The space of ECH index 0 curves is not necessarily compact.
- It's just not the right thing to count. There are contributions from negative ECH index curves due to multiple covers.

Both of these problems arise from the existence of negative ECH index multiply covered curves. In the cylindrical case, we can eliminate these curves by turning them into multiple simple curves by translation. However, this trick is no longer available to us.
It's easiest to work in the case of cobordism that arise from changing the contact structure. We visualize this by looking at the return map around the Reeb orbit.

3.1 Where there are no ECH index 0 curves, but should be a count

In this example we explore how an elliptic orbit turns into a negative hyperbolic orbit and an elliptic orbit of double the period.
The deformation in terms of the linearized return map can be thought of as a path through the $Sp(2)$. If the path passes through the Maslov cycle, we will get some interesting results.
The picture is this– deform the contact structure so that the rotational return map goes into rotation by π. Then this is a hyperbolic orbit, and then let one of the parameters grow and shrink. Then this is not one of our maps unless we consider , because going around the orbit twice is degenerate.

4 How we define the cobordism map

For the above reasons, we have a difficult counting the number of index 0 towers that go in between Reeb orbits. In order to define the cobordism map, we go over to SWF, and use the cobordism map from there. Taubes proved that there is a canonical isomorphism between Seiberg-Witten Floer theory and ECH.

Theorem 4.1 Fix a Spin_c structure on M. Then there is an isomorphism between embedded contact homology and Seiberg-Witten homology. Furthermore, this isomorphism respects the cobordism.

While this isomorphism doesn't actually give us a count of the towers that show up in the in the cobordism. However, we are able to recover the following:

Theorem 4.2 Suppose that the cobordism map ϕ from SWT is nonzero on an element in the sense that $\langle \phi\alpha, \beta\rangle \neq 0$. Then there exists a broken holomorphic curve between α and β in the cobordism.

4.1 Some of the ideas behind SWF to ECH

Here are some of the ideas behind the isomorphism between SWF and ECH. Here is an outline to the SW invariants. Seiberg-Witten Floer theory looks at critical points of the Seiberg-Witten action, which is dependent on a perturbation term, and a choice of spin structure.

- The space of spin structures are identified with $H^2(X, \mathbb{Z})$ by the symplectic structure on X.
- As the perturbation term is increased, the solutions converge as a current to a J-holomorphic current.