Lagrangian Cobordism and Surgery

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Summary

Introduction

Inspiration: Topology of Manifolds

- Two manifolds $M^{k,n-k,+}, M^{k,n-k,-}$ are related by k- surgery if there exists a manifold M with boundary $\partial M = S^k \times S^{n-k-1}$ from which

$$\begin{aligned} M^{k,n-k,+} =& M \cup (S^k \times D^{n-k}) \\ M^{k,n-k,-} =& M \cup (D^{k+1} \times S^{n-k-1}) \end{aligned}$$

- There exists a surgery trace cobordism

$$N^{k,n-k+1} : M^{k,n-k,+} \rightsquigarrow M^{k,n-k,-}.$$

- Invertible: $N^{k,n-k+1} = (N^{n-k,k+1})^{-1}$

Inspiration: Topology of Manifolds

Theorem

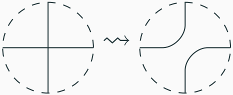
Every cobordism $N : M^+ \rightsquigarrow M^-$ can decomposed into cobordisms of the form

$$N = N^{k_1, n-k_1+1} \circ \cdots \circ N^{k_j, n-k_j+1}$$

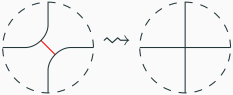
Lagrangian Surgeries

We can also perform surgery on Lagrangian submanifolds.

- Polterovich 1991 connect sum



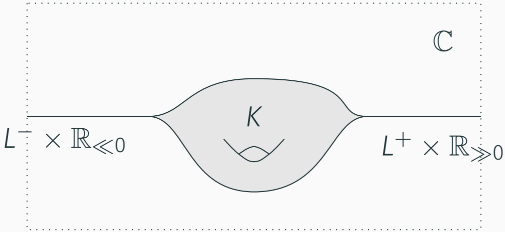
- Audin, Lalonde, and Polterovich 1994,Dimitroglou Rizell 2016 and Haug 2020 k-antisurgeries



Lagrangian Cobordisms

Definition (Arnol'd 1980)

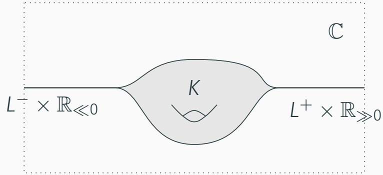
Let L^- and L^+ be Lagrangian submanifolds of X. A Lagrangian cobordism $K : L^+ \rightsquigarrow L^-$ is a Lagrangian in $X \times \mathbb{C}$ with ends limiting to $L^- \times \mathbb{R}_{\ll 0}$ and $L^+ \times \mathbb{R}_{\gg 0}$.

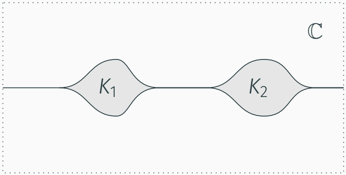


Results: Geometry of Lagrangian Cobordisms

Theorem (H)

Let $K : L^+ \rightsquigarrow L^-$ be a Lagrangian cobordism. K is exactly homotopic to the concatenation of surgery trace cobordisms and suspensions of exact homotopies.





Why Use Lagrangian Cobordisms?

- Work of Biran and Cornea; Nadler and Tanaka show that Lagrangian cobordisms correspond to iterated exact sequences in the Fukaya category.
- As a specialization: if $K : L^+ \rightsquigarrow L^-$ is a monotone Lagrangian cobordism, then:

$$0 \to L^+ \to L^- \to 0.$$

- Proof is non-constructive.

Consequences of Surgery Decompositions

We hope that our decomposition will allow us to:

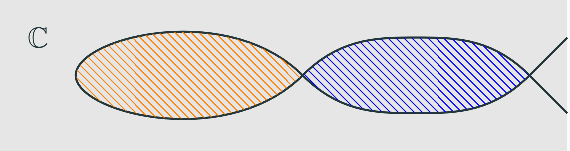
- Compute the equivalences induced by Lagrangian cobordisms from surgery decomposition.
- Extend Biran and Corena's work to *unobstructed* Lagrangian cobordisms.

This requires an understanding of the holomorphic disks on Lagrangian surgery traces.

Results: Floer Theory of Surgery Traces

Theorem

Let $K^{k,n-k+1}$ be the local model of the Lagrangian surgery trace. There exists a holomorphic teardrop with boundary on $K^{k,n-k+1}$ pairing the generators on Floer cohomology arising from the Morse critical point of $\pi_{\mathbb{R}} : K \to \mathbb{R}$ and self-intersection point.



Background: Lagrangian Cobordism

Example I: Suspension of Exact Homotopy

- Let $i_t : L \to X$ be an exact Lagrangian Homotopy
- Primitive $H_t : L \to \mathbb{R}$ satisfying $dH_t = i_t^* \omega \left(\frac{di_t}{dt}, - \right)$.

The suspension cobordism

$$\begin{aligned} L \times \mathbb{R} &\hookrightarrow X \times \mathbb{C} \\ (x, t) &\mapsto (i_t(x), t + \imath H_t(x)) \end{aligned}$$

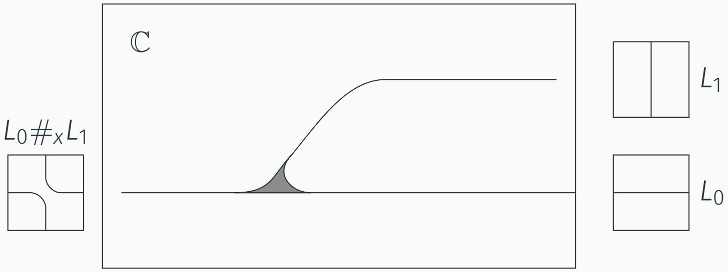
is a Lagrangian cobordism between $i_0(L)$ and $i_1(L)$.

Example II: Connect Sum and Cobordism

Definition

Let L^0, L^1 be Lagrangian submanifolds with transverse intersection at a point x . There is a *surgery cobordism* $K : (L_0, L_1) \rightsquigarrow L_0\#_x L_1$.

Relates to surgery triangle from Fukaya, Oh, Ohta, and Ono 2007



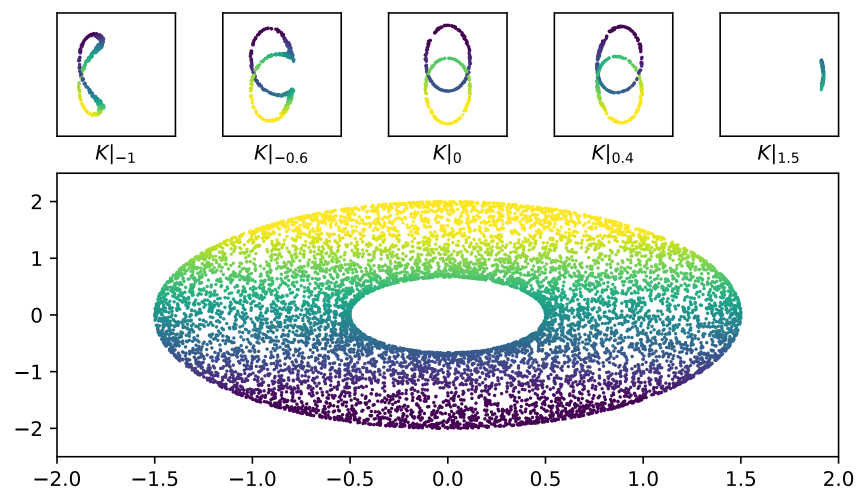
Visualizing Cobordisms: Slice and Shadow

Let $K \subset X \times \mathbb{C}$ be a Lagrangian cobordism. The *slice* of the cobordism at time t is

$$K|_t := \pi_X(\pi_{\mathbb{R}}^{-1}(t)) \subset X.$$

When t is a regular value of the projection to the real coordinate of $\mathbb{C}$, then $K|_t$ is an immersed Lagrangian submanifold of X.

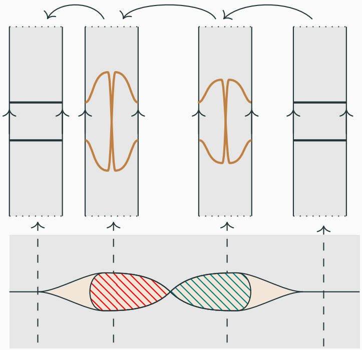
Example: Product Torus in $\mathbb{C} \times \mathbb{C}$.



Example: Equivalence from Cobordism

(a) Slices

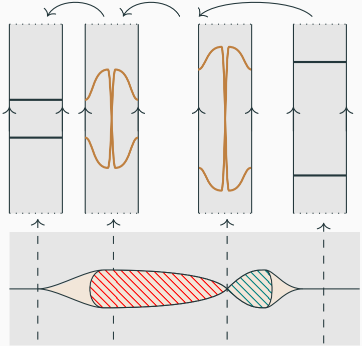
(b) Shadow



Example: Non-Equivalence from Cobordism

(a) Slices

(b) Shadow



Lagrangian Surgery Traces

Null cobordant Spheres

Definition

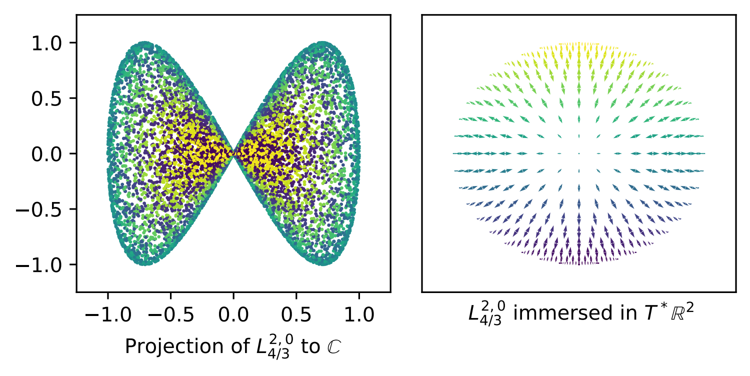
The Whitney sphere of area A is the Lagrangian submanifold $L_A^{n,0,+} \subset \mathbb{C}^n$ which is parameterized by

$$\begin{aligned} i_A^{n,0,+} : S_r^n \to & \mathbb{C}^n \\ (x_0, \ldots, x_n) \mapsto & (x_1 + \jmath 2x_0x_1, x_2 + \jmath 2x_0x_2, \ldots, x_n + \jmath 2x_0x_n). \end{aligned}$$

where $S^n = \{(x_0, \ldots x_n) \mid \sum_{i=0}^n x_i^2 = r^2\}$, and $r = \sqrt[3]{\frac{4}{3}A}$.

The $+$ is supposed to help us remember that this is immersed.

Picture: Whitney Sphere



Bounds a holomorphic teardrop of area $A = 4/3$

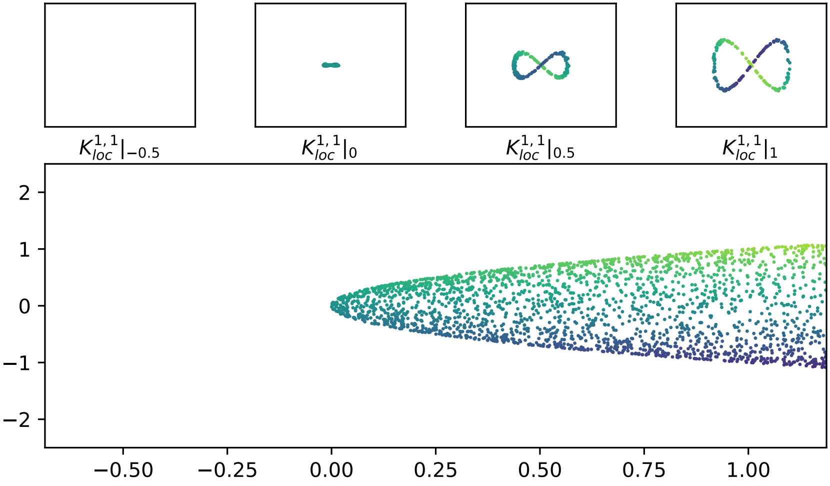
Null cobordant Spheres

Decreasing the area A is an exact isotopy of Lagrangian submanifolds. The parameterization

$$\begin{aligned} j^{n,1} : \mathbb{R}^{n+1} &\to \mathbb{C}^n \times \mathbb{C} \\ (x_0, \dots, x_n) &\mapsto (i_r^{n,0}(x_0, \dots, x_n), r^2 - \jmath x_0). \end{aligned}$$

gives an embedded Lagrangian disk $K^{n,1} \subset \mathbb{C}^n \times \mathbb{C}$

Picture: Null Cobordism of Whitney Sphere



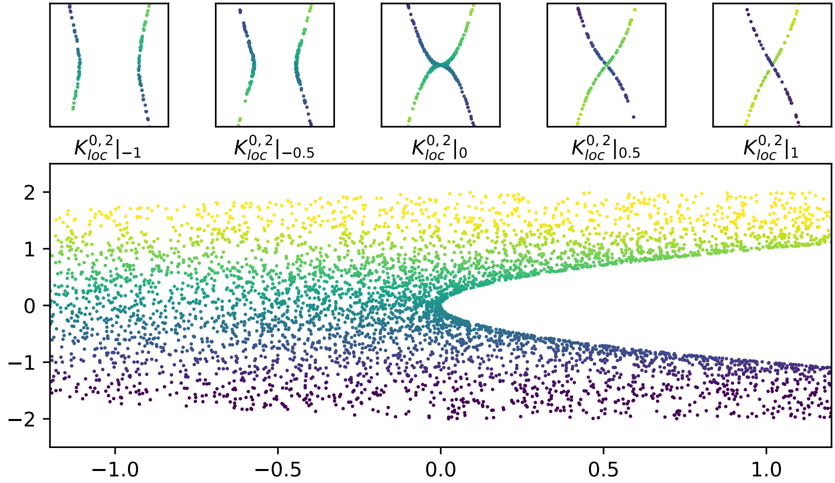
Lagrangian $(k, n-k+1)$ surgery trace

Definition (Audin, Lalonde, and Polterovich 1994)

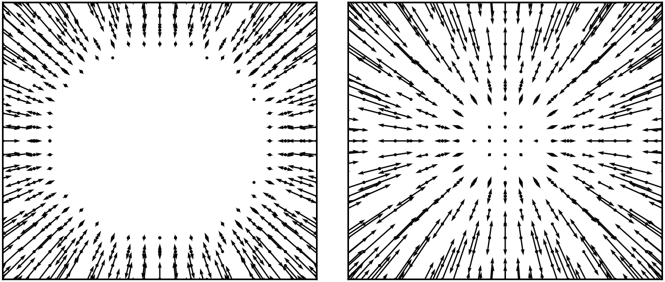
For $k \geq 0$, the local Lagrangian $(k, n-k+1)$ surgery trace is the Lagrangian submanifold $K_{loc}^{k,n-k+1} \subset (\mathbb{C})^n \times \mathbb{C}$ parameterized by

$$\begin{aligned} j^{k,n-k+1} : \mathbb{R}^{n+1} &\to T^*\mathbb{R}^n \times \mathbb{C} \\ (x_0, x_1, \ldots, x_n) &\mapsto \left(x_1 + \jmath\sigma_{1,k}2x_1x_0, \ldots, x_n + \jmath\sigma_{n,k}2x_nx_0, x_0^2 + \textstyle\sum_{i=1}^n \sigma_{i,k}x_i^2 - \jmath x_0\right). \end{aligned}$$

Picture: $(0,2)$ surgery trace

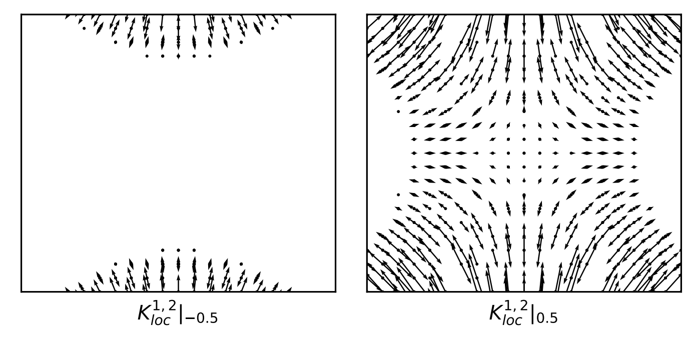


Picture: $(0,3)$ surgery: before and after



$K^{0,3}_{loc}|_{-0.5}$ $K^{0,3}_{loc}|_{0.5}$

Picture: $(1,2)$ surgery: before and after



Some immediate observations

The positive and negative slices of this Lagrangian are denoted

$$L_{loc}^{k,n-k,+} := K_{loc}^{k,n-k+1}|_1 \qquad L_{loc}^{k+1,n-k,-} := K^{k,n-k+1}|_{-1}.$$

- In first k-coordinates, positive end contains an Whitney sphere

$$L_{loc}^{k,n-k,+}|_{\mathbb{C}^k} = L_{loc}^{k,0,+} \subset \mathbb{C}^k$$

- In first k-coordinates, negative end is empty.

$$L_{loc}^{k,n-k,-}|_{\mathbb{C}^k} = \emptyset.$$

Important Take away

Unlike cobordisms for manifolds, the inverse of a surgery trace *is not another surgery trace!*

Decomposing Lagrangian Cobordisms into Surgery Traces

Theorem (H)

Let $K : L^+ \rightsquigarrow L^-$ be a Lagrangian cobordism. Then there is a sequence of Lagrangian cobordisms

$$K_{H_t^i} : L^-_{i+1} \rightsquigarrow L^+_i \text{ for } i \in \{0, \ldots, j\}$$
$$K_i^{k_i, n-k_i+1} : L^+_i \rightsquigarrow L^-_i \text{ for } i \in \{1, \ldots j\}$$

which satisfy the following properties:

- $L^-_{j+1} = L^+$ *and* $L^+_0 = L^-$
- *Each* $K^{k_i, n-k_i+1}$ *is a Lagrangian surgery trace;*
- *Each* $K_{H_t^i}$ *is the suspension of an exact homotopy and;*
- *There is an exact homotopy between*

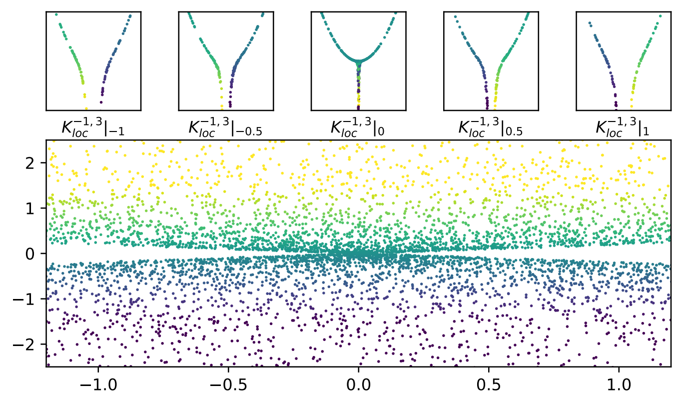
$$K \sim K_{H_t^j} \circ K_j^{k_j, n-k_j+1} \circ K_{H_t^{j-1}} \circ \cdots \circ K_{H_t^1} \circ K_1^{k_1, n-k_1+1} \circ K_{H_t^0}.$$

Sketch of Proof

1. Apply exact homotopy so that $\pi_{\mathbb{R}} : K \to \mathbb{R}$ is a Morse function.
2. Decompose K into cobordisms $K_1 \circ \cdots \circ K_j$, each containing a single critical point of $\pi_{\mathbb{R}}$.
3. Show that each of these pieces is homotopic to a surgery trace.

“Good position” for Lagrangian cobordisms

Only the third step is problematic; need to make level set of $\pi_{j\mathbb{R}}$ to transverse to critical level set of $\pi_{\mathbb{R}}$.



Applications

Self Intersection and Euler Characteristic

Each Lagrangian surgery modifies the Lagrangian by removing a self intersection, and adding a surgery handle.

- Denote the self intersection of a Lagrangian submanifold L by

$$\mathcal{I}^{si}(L) = \{(p \to q) \mid p, q \in L, i(p) = i(q)\}$$

- Denote the Morse critical points of a pair (L, f) by

$$\mathcal{I}^{f}(L) = \text{crit}(f)$$

- Let $\mathcal{I}(L) = \mathcal{I}^{si}(L) \cup \mathcal{I}^{f}(L)$.

Invariance of Euler Characteristic

When L is graded, write

$$\chi^{si}(L) := \sum_{x \in \mathcal{I}^f \cup \mathcal{I}^{si}} (-1)^{\mathrm{ind}(x)}$$

Claim

If $K : L^+ \rightsquigarrow L^-$ is a graded Lagrangian cobordism, $\chi^{si}(L^+) = \chi^{si}(L^-)$

This can be proven directly, but we use the decomposition to prove this by checking on surgery traces.

Surgery Preserves χ^{si}

Let $K^{k,n-k+1} : L^+ \rightsquigarrow L^-$ be a surgery cobordism. Then for appropriate choices of Morse function, $\mathcal{I}(L^\pm)$ differ at the following elements.

Index	$\mathcal{I}^{si}(L^+)$	$\mathcal{I}^{f^+}(L^+)$	$\mathcal{I}^{f^-}(L^-)$
k		x^+	
$k+1$	$(q_+ \to q_-)$		
$n-k-1$	$(q_- \to q_+)$		x^-

Next steps

Does this argument for Euler characteristic extend to immersed Lagrangian Floer cohomology?

Definition (Sketch)

Let L be an immersed Lagrangian submanifold. The immersed Lagrangian Floer cohomology of L is has:

- Chains defined by $CF^\bullet(L) := \Lambda\langle\mathcal{I}(L)\rangle$
- Differential which counts configurations of treed disks and teardrops.

This is a filtered A_∞ algebra (m^0 term from disks and teardrops).

Questions about Floer Cohomology and Cobordisms

Question (Unobstructedness Problem)

When is $CF^\bullet(K)$ unobstructed by bounding cochain?

Question (Continuation Problem)

If $CF^\bullet(K, b)$ is unobstructed by bounding cochain, are $CF^\bullet(L^-, b^-)$ and $CF^\bullet(L^+, b^+)$ A_∞ homotopic?

We hope to provide a criteria using the decomposition of $CF^\bullet(K, b)$ into surgery traces.

Floer Cohomology of Cobordisms

An expected property of a Lagrangian cobordism $K : L^+ \rightsquigarrow L^-$ is that its Floer cohomology has projections to the Floer cohomology of the ends.

$$CF^\bullet(L^-) \xleftarrow{\pi^-} CF^\bullet(K) \xrightarrow{\pi^+} CF^\bullet(L^+)$$

We can prove that $CF^\bullet(L^-, b^-)$ and $CF^\bullet(L^+, b^+)$ are A_∞ homotopic if this is a mapping cylinder for the identity.

Unobstruct. and Cont. from Decomposition?

We first apply our geometric decomposition.

$$CF^\bullet(L_0) \xleftarrow{\pi^-} CF^\bullet(K_0) \xrightarrow{\pi^+} CF^\bullet(L_1) \xleftarrow{\pi^-} CF^\bullet(K_1) \xrightarrow{\pi^+} \cdots \xleftarrow{\pi^-} CF^\bullet(K_j) \xrightarrow{\pi^+} CF^\bullet(L_j)$$

where each K_i is a surgery trace of exact homotopy.

Conjecture (Unobstructedness Problem)

$CF^\bullet(K)$ is unobstructed if each K_j can be equipped with a bounding cochain with compatible restrictions.

Conjecture (Continution Problem)

If K_j is unobstructed, it is a mapping cylinder.

Floer Cohomology of Surgery Trace I

The surgery trace $K^{k,n-k+1}$ has a standard immersed form which has

- Distinguished self intersection points corresponding to the surgery point and
- Distinguished critical point of $\pi_{\mathbb{R}} : K \to \mathbb{R}$ corresponding to the surgery handle.

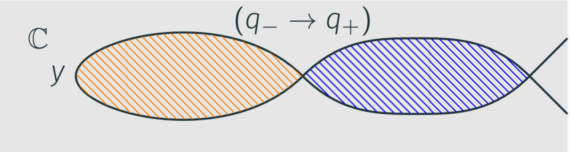
Theorem (Unobstructedness Problem)

If $k = 0$, there is a Maslov index 0 regular holomorphic teardrop with output on the distinguished self-intersection.

Theorem (Continuation Problem)

There exists a holomorphic teardrop with boundary on the surgery trace exactly pairing the distinguished self-intersection and critical point in $CF^\bullet(K)$.

Distinguished Generators and Holomorphic Teardrops



Index

$k+1$ — m^0 if $k = 0$ —T^A→ ; $(q_+ \to q_-)_+$ —T^B→

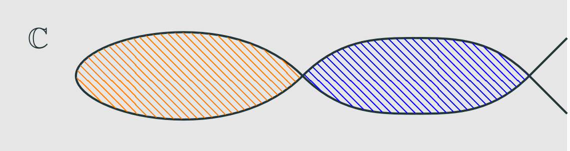
$k+2$ — $(q_+ \to q_-)_-$

$n-k$ — y ←T^A—

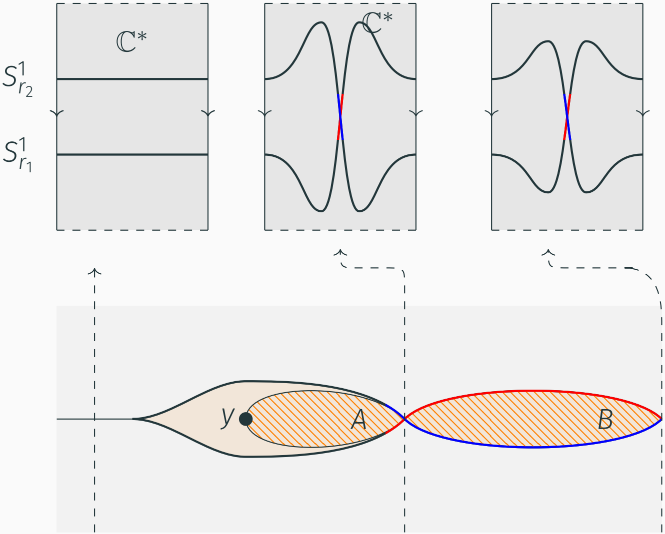
$n-k-1$ — $(q_- \to q_+)_-$

Unobstructedness Problem in Pictures

To first order, the Maurer-Cartan equation for a Lagrangian surgery trace cobordism is determined by the neck size of the surgery.



Continuation Problem in Pictures



Further Directions

Non-isotopic, Floer equivalent Lagrangians

A natural continuation point is to use these techniques to construct interesting equivalences of objects in the Fukaya category.

Question

Do there exist non-isotopic monotone cobordant Lagrangian submanifolds?

In previous work , we showed that unobstructed Lagrangian cobordisms provided equivalences related to wall-crossing and mutation.

Structure of the Cobordism Group

Biran and Cornea conjecture that the Lagrangian cobordism group is exactly the Grothendieck group of the Fukaya category.

Question

Does the decomposition prove the only equivalences in the Lagrangian cobordism group are K-theoretic?

Lefschetz Fibrations

All of the techniques applied appear to work in the setting of Lefschetz fibrations, conjecturally providing a decomposition result for Lagrangian submanifolds into a collection of surgery traces of thimbles.

Question

Does this decomposition provide a geometric interpretation to generation of the Fukaya Seidel Category?

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