# Generating the Mirror to a Toric Variety

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# Summary

Piecewise Linear Picture

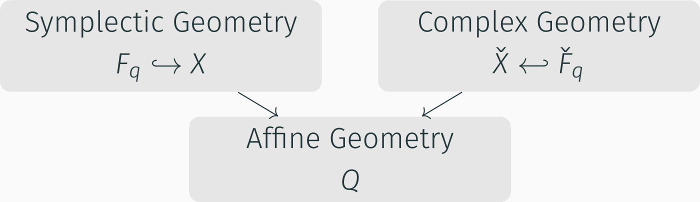
Lagrangian Correspondence and Cobordism

Generation of FLTZ Linking Disks

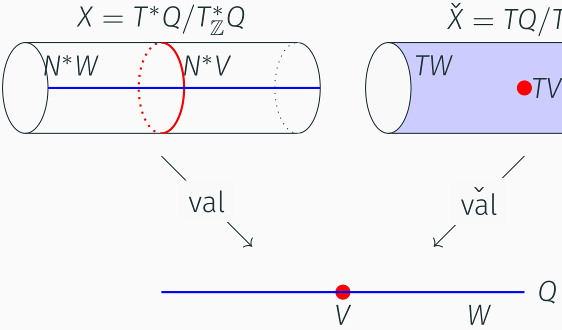
# Some Motivation:

Mirror Symmetry proposes that there exists a dictionary between symplectic geometry and complex geometry.

- Symplectic spaces $X$ and "mirror" complex spaces $\check{X}$ have dual Lagrangian torus fibrations.
- Use this dual Lagrangian torus fibration to show geometry on $X, \check{X}$ is interchanged



# Construction of Mirror Spaces



## Matching Invariants

Homological Mirror Symmetry gives a precise statement of this dictionary:

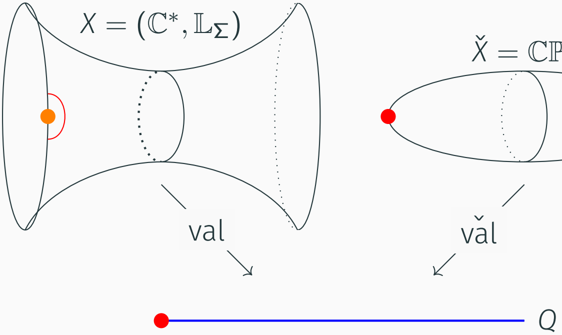
*A-model on $X$*

- Lagrangian Submanifolds
- Floer Cohomology
- Fukaya Category
- "Stops"

*B-model on $\check{X}$*

- Coherent Sheaves
- Sheaf Ext
- $D^b\mathrm{Coh}(X)$
- Compactification

# Example: Mirror to $\mathbb{CP}^1$



# HMS General Strategy

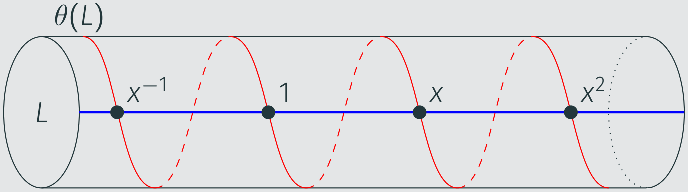
1. Construct mirror functor between some Lagrangians and sheaves

$$\{L_i\} \leftrightarrow \{F_i\}.$$

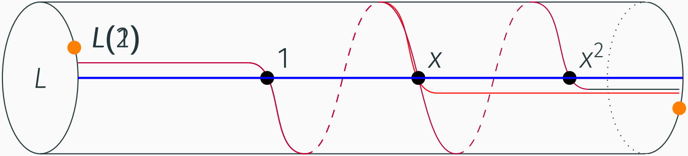
2. Show that $\{L_i\}$ generate $\mathrm{Fuk}(X)$, and $\{F_i\}$ generate $D^b\,\mathrm{Coh}(\check{X})$.

## Example

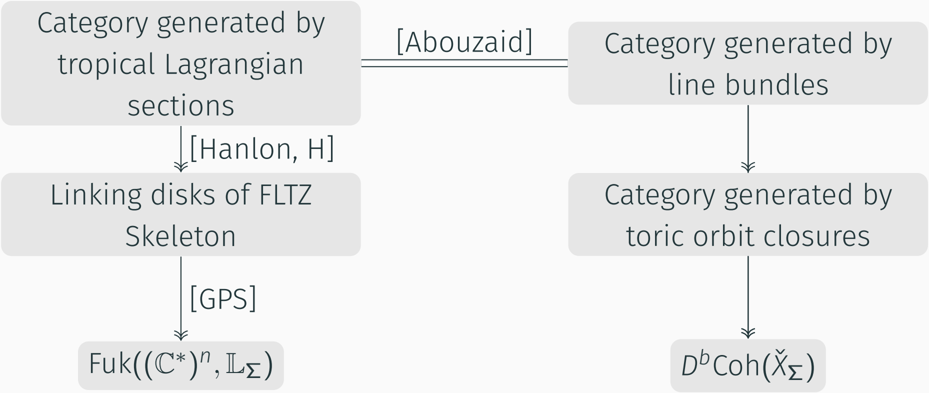
Wrapped category of $T^*S^1$ is generated by a fiber:



# Mirror Symmetry for $\mathbb{CP}^1$



# Example: Mirror Symmetry for toric Varieties



# Statement of Results

**Theorem (Joint with A. Hanlon)**

*Lagrangian sections generate the linking disks of the FLTZ skeleton.*

**Theorem (Ganatra, Pardon, and Shende 2018)**

*Linking disks and cocores generate the partially wrapped Fukaya category.*

**Corollary**

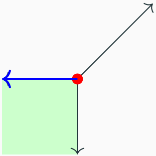
$\mathrm{Fuk}((\mathbb{C}^*)^n, \mathbb{L}_\Sigma)$ *is derived equivalent to* $D^b\,\mathrm{Coh}(X_\Sigma)$.

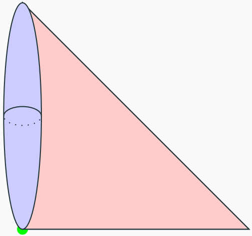
# Piecewise Linear Picture

# Notation for Toric Varieties

Let $\check{X}_\Sigma$ be a projective toric variety.

- Comes with Lagrangian torus fibration (moment map.)
- Orbits of action are indexed by cones of fan $\Sigma$.





# FLTZ Skeleton

Each boundary component of moment polytope — equivalently cone of $\sigma \in \Sigma$ — gives a stop

$$\mathbb{L}_\sigma := \sigma \times \sigma^\perp \subset \mathbb{R}^n \times T^n.$$

**Definition**

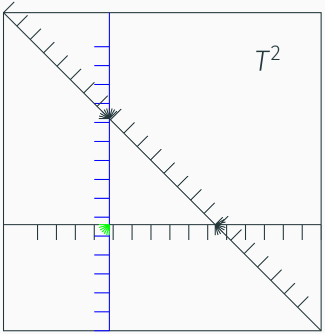
Let $\Sigma$ be a fan. Then Fang, Liu, Treumann, and Zaslow 2012-skeleton is the Lagrangian

$$\mathbb{L}_{\boldsymbol{\Sigma}} = \bigcup_{\sigma \in \boldsymbol{\Sigma}} \mathbb{L}_\alpha \in \mathbb{R}^n \times T^n$$

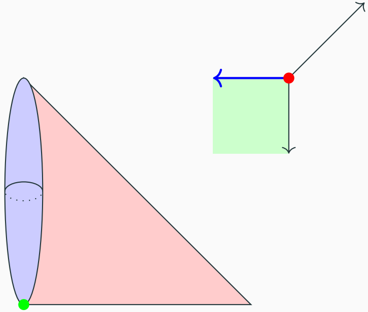
# Drawing the FLTZ Skeleton for $\mathbb{CP}^2$

$$\mathbb{L}_\Sigma = \bigcup_{\sigma \in \Sigma} \sigma \times \sigma^\perp \in \mathbb{R}^n \times T^n$$

$(\mathbb{C}^*)^2 = T^*T^2$



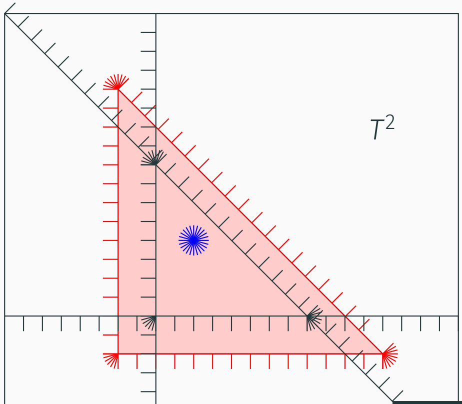

$\check{X}_\Sigma$



# Plotting Lagrangian Sections

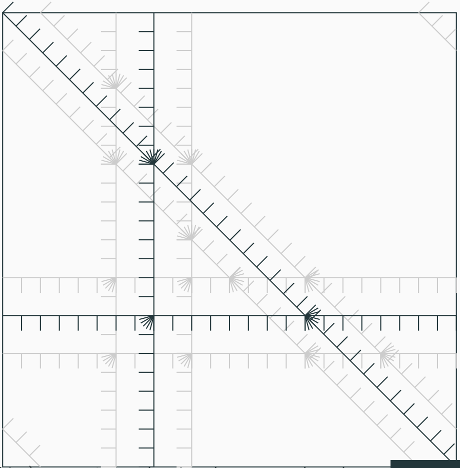
Because Hamiltonian isotopies are now restricted so that they avoid the stop, sections of $(\mathbb{C}^*)^2 \to \mathbb{R}^2$ are no longer admissibly isotopic.

$$(\mathbb{C}^*)^2 = T^*T^2$$



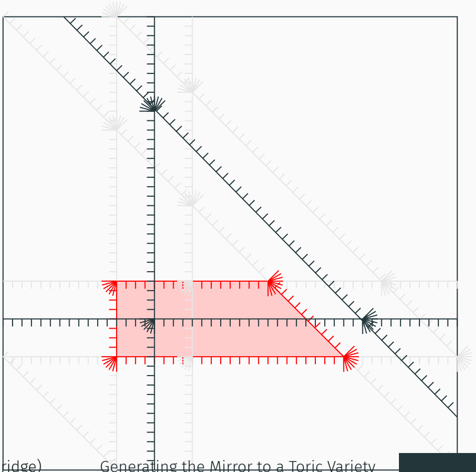
# A Collection of PL Lagrangians

We now describe a collection of PL Lagrangians with boundary on a shift of the FLTZ skeleton.



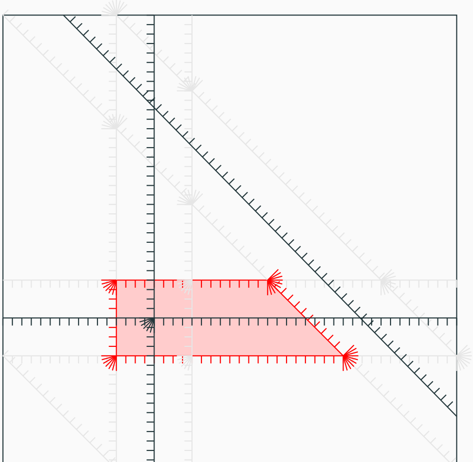
# Intersection with FLTZ Skeleton

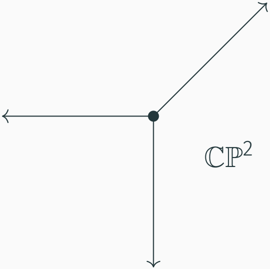
It is easy to read off the intersections between PL Lagrangian disks and FLTZ skeleton after taking a perturbation.



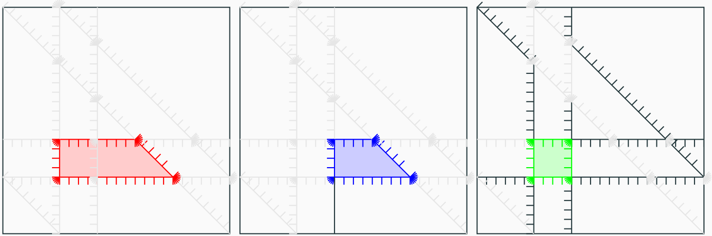
# Relation to Fan and Mirror Symmetry:

Each PL Lagrangian disk corresponds to the pushforward of a line bundle from a toric orbit.





# Symmetric Difference



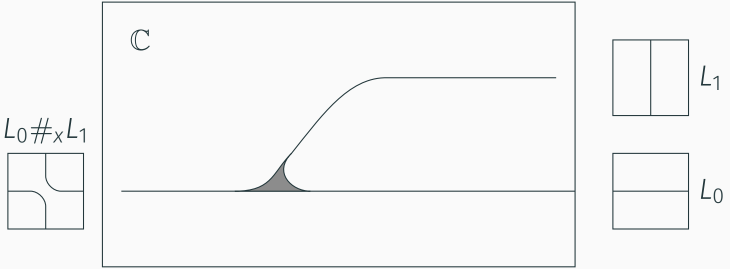
$$(\hat{L}_\sigma(\phi) \cup \hat{L}_\sigma(\phi')) \setminus (\hat{L}_\sigma(\phi) \cap \hat{L}_\sigma(\phi')) = \hat{L}_\tau(\phi')$$

Gives a Lagrangian cobordism.

# Lagrangian Cobordism

## Definition (Arnol'd 1980)

Let $L^1, \ldots, L^k$ and $L^0$ be Lagrangian submanifolds of $X$. A Lagrangian cobordism $K : (L^i) \rightsquigarrow L^0$ is a Lagrangian in $X \times \mathbb{C}$ with ends limiting to $L^j \times \{\imath c_j + \mathbb{R}_{\ll 0}\}$ and $L^0 \times \mathbb{R}_{\gg 0}$.



# Lagrangian Cobordism and Generation

**Theorem (Biran and Cornea 2014)**

*If there exist a monotone Lagrangian cobordism $K : (L^i) \rightsquigarrow L^0$, then $(L^i)$ generate $L^0$ in the Fukaya category.*

Current work of Valentine Bosshard extending this to the partially wrapped setting.

# Summary:

We've associated to toric orbit closure $V(\sigma) \subset X_\Sigma$, and divisors $\phi$ a PL Lagrangian disk $L_\sigma(\phi)$ of $((\mathbb{C}^*)^n, \mathbb{L}_\Sigma)$.

**Linking Disk Property** These PL Lagrangian disks have easy to characterize intersection with FLTZ skeleton.

**Cobordism Property** There is a Lagrangian cobordism between these disks (which implies generation)

Want to move the entire story over to smooth Lagrangians.

# Lagrangian Correspondence and Cobordism

# Lagrangian correspondences

Let $X_1, X_2$ be symplectic manifolds. A *Lagrangian correspondence*

$$L_{12} : X_1 \Rightarrow X_2$$

is a Lagrangian submanifold $L_{12} \subset (X_1, -\omega_1) \times (X_2, \omega_2)$.

**Example (Symplectomorphism)**

Given a symplectomorphism $f : X_1 \to X_2$ the graph $\Gamma(f) \subset X_1^- \times X_2$ is a Lagrangian correspondence.

# Geometric Composition

Given Lagrangian correspondences

$$L_{12} : X_1 \Rightarrow X_2 \qquad\qquad L_{23} : X_2 \Rightarrow X_3$$

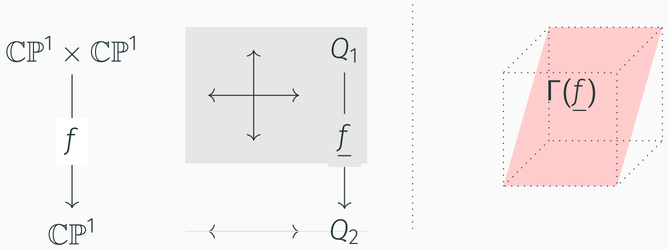
there is a geometric composition:

$$L_{23} \circ L_{12} := \{(x_1, x_3) \mid (x_1, x_2) \in L_{12}, (x_2, x_3) \in L_{23} \subset X_1^- \times X_3$$

If this is cut out transversely, it is again a Lagrangian correspondence.

# An Example

Let $f : \check{X}_{\Sigma_1} \to \check{X}_{\Sigma_2}$ be a toric morphism. There is an associated linear map $\underline{f} : Q_1 \to Q_2$, giving us a graph $\Gamma(\underline{f}) \subset Q_1 \times Q_2$. There exists a Lagrangian lifting $N^*\Gamma(\underline{f})/N^*_{\mathbb{Z}}\Gamma(\underline{f}) \subset X_1 \times X_2$.



# Lagrangian correspondence and Lagrangian Cobordism

**Lemma**

*Suppose that* $K_{12} : (L^i_{12}) \rightsquigarrow (L^0_{12})$ *is a Lagrangian cobordism of Lagrangian correspondences in* $X_1^- \times X_2 \times \mathbb{C}$ *Let* $L_{23} : X_2^- \times X_3$ *be another Lagrangian correspondence. Then there exists a Lagrangian cobordism*

$$L_{23} \circ K_{12} : (L_{23} \circ L^i_{12}) \rightsquigarrow (L_{23} \circ L^0_{12})$$

# Idea of Proof: Inclusion of Orbit

**Proposition**

*Given $\alpha \in \Sigma$ a 1-dimensional cone, there exists a Lagrangian correspondence $L_{\alpha,0} \subset ((\mathbb{C}^*)^{n-1}, \mathbb{L}_{V(\alpha)}) \Rightarrow ((\mathbb{C}^*)^n, \mathbb{L}_{\mathbf{\Sigma}})$ with the following properties:*

- *If $L_\alpha(\phi) \subset ((\mathbb{C}^*)^{n-1}, \mathbb{L}_{V(\alpha)})$ is a section, then there exists sections $L_0(\phi), L_0(\phi - \alpha) \in ((\mathbb{C}^*)^n, \mathbb{L}_{\mathbf{\Sigma}})$ and a Lagrangian cobordism,*

$$(L_0(\phi - \alpha), L_0(\phi)) \rightsquigarrow L_{\alpha,0} \circ L_\alpha(\phi)$$

- *Sends linking disks of $\mathbb{L}_{V(\alpha)}$ to linking disks of $\mathbb{L}_{\mathbf{\Sigma}}$.*

# Construction of the Inclusion

Idea of construction is easiest to follow from $B$-model construction of $\Gamma(i) : \check{V}(\alpha) \to \check{X}_\Sigma$. Assume for simplicity that $\check{X}_\Sigma = N_{\check{X}_\Sigma}\check{V}(\alpha)$.

1. There exists a toric projection $\pi : \check{X}_\Sigma \to V(\alpha)$.
2. The map of $i : V(\alpha) \to X_\Sigma$ is not toric, but we have an exact sequence:

$$\mathcal{O}_{\Gamma(\pi)} \otimes \mathcal{O}(-\alpha) \to \mathcal{O}_{\Gamma(\pi)} \to \mathcal{O}_{\Gamma(i)}$$

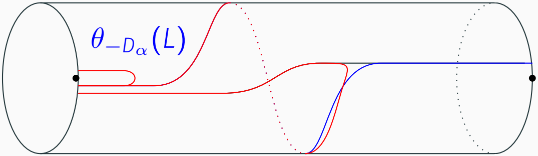
We recreate this construction on the $A$-side using *twisting Hamiltonian* and *Lagrangian Cobordism*.

# Twisting Hamiltonian + Surgery

Hanlon 2019 constructs a twisting Hamiltonian associated to a divisor $D \subset X_\Sigma$

Hicks 2019 a surgery for Lagrangians with collared overlaps

$$\mathbb{C}^* = T^*\mathbb{R}/T^*_{\mathbb{Z}}\mathbb{R}$$



# General Construction of $L_{\alpha,0}$

Again, assume $\check{X}_\Sigma = N_\Sigma V(\alpha)$, so we have a toric map $\pi : X_\Sigma \to V(\alpha)$.

1. From before, we have Lagrangian correspondence $L_{\underline{\pi}} : ((\mathbb{C}^*)^n, \mathbb{L}_\Sigma) \Rightarrow ((\mathbb{C}^*)^{n-1}, \mathbb{L}_{V(\alpha)})$.
2. The inclusion correspondence is defined by surgery:

$$L_{\alpha,0} := \theta_\alpha^{-1}(L_{\underline{\pi}})\#L_{\underline{\pi}}$$

$$\mathcal{O}_{\Gamma(i)} = \text{cone}(\mathcal{O}_{\Gamma(\pi)} \otimes \mathcal{O}(-\alpha) \to \mathcal{O}_{\Gamma(\pi)})$$

# Property of $L_{\alpha,0}$

Let $L_\alpha(\phi) \subset ((\mathbb{C}^*)^{n-1}, \mathbb{L}_{V(\alpha)})$ be a Lagrangian section. Then $L_{\underline{\pi}}^{-1} \circ L_\alpha$ is a Lagrangian section of $((\mathbb{C}^*)^n, \mathbb{L}_{\Sigma})$. Obtain a Lagrangian cobordism

$$\begin{aligned}
(\theta_\alpha^{-1}(L_{\underline{\pi}}), L_{\underline{\pi}}) &\rightsquigarrow L_{\alpha,0} \\
(\theta_\alpha^{-1}(L_{\underline{\pi}}) \circ L_\alpha(\phi), L(\underline{\pi}) \circ L_\alpha(\phi)) &\rightsquigarrow L_{\alpha,0} \circ L_\alpha(\phi) \\
(L_0(\phi - \alpha), L_0(\phi)) &\rightsquigarrow L_{\alpha,0} \circ L_\alpha(\phi)
\end{aligned}$$

# Generation of FLTZ Linking Disks

# Linking Disks

A linking disk of $\mathbb{L}_{\Sigma}$ is a disk which intersects the stop at a single point.

## Claim

*$L_{\alpha,0} : ((\mathbb{C}^*)^{n-1}, \mathbb{L}_{V(\alpha)}) \Rightarrow ((\mathbb{C}^*)^n, \mathbb{L}_{\Sigma})$ sends core and linking disks of $\mathbb{L}_{V(\alpha)}$ to linking disks of $\mathbb{L}_{\Sigma}$.*

## Proof.

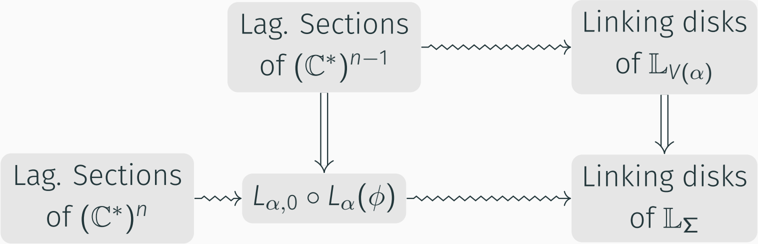
Main components of proof:

- $L_{\alpha,0} \circ L_\alpha$ has the topology of $L_\alpha \times \mathbb{R}$.
- $L_{\alpha,0} \circ \mathbb{L}_{V(\alpha)} \subset \mathbb{L}_{\Sigma}$.
- Also need to check that every section $L_\alpha(\phi)$ has intersection with the FLTZ skeleton.

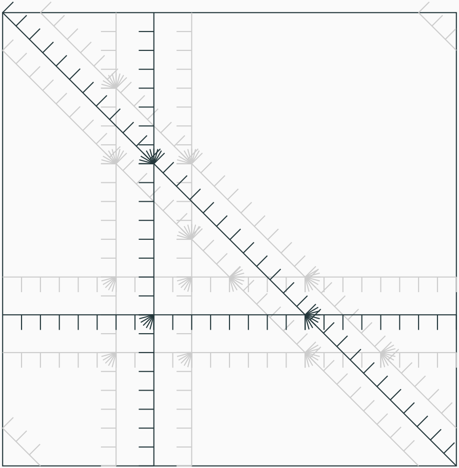
□

# Generation of Linking Disks

It follows if Lagrangian sections generate linking disks for $((\mathbb{C}^*)^{n-1}, \mathbb{L}_{V(\alpha)})$ , then Lagrangian sections generate linking disks on $((\mathbb{C}^*)^n, \mathbb{L}_\Sigma)$.



# Example: Generating $\mathbb{CP}^2$



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# Discussion