Realizability and Obstruction in Tropical Geometry

Jeff Hicks

2021

University of Cambridge

Summary

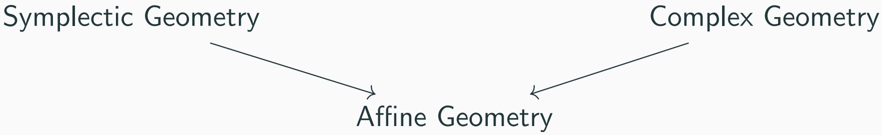
Background: Mirror Symmetry

Mirror Symmetry

Mirror Symmetry is a duality between symplectic geometry on a manifold X and complex geometry on a *mirror manifold* $\check{X}$.

- What is the mechanism which interchanges these two types of manifolds?
- What portions of the geometry are interchanged?

Symplectic $\leftrightarrow$ Affine $\leftrightarrow$ Complex



From Affine to ω and J

Definition

An integral affine manifold Q is a manifold possessing charts whose transitions are all integral affine transformations.

Main feature: Lattices inside of (co)tangent bundle.

$$T^*_{\mathbb{Z}}Q \subset T^*Q$$
$$T_{\mathbb{Z}}Q \subset TQ$$

Building the complex and symplectic manifolds

Remark

T^*Q has a symplectic structure. TQ has an almost complex structure.

$$(X, \omega) := (T^*Q/T^*_{\mathbb{Z}}Q, dp \wedge dq).$$

$$(\check{X}, J) := (TQ/T_{\mathbb{Z}}Q, J).$$

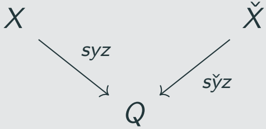
Example (Running Example)

If $Q = \mathbb{R}^n$, then $X = \check{X} = (\mathbb{C}^*)^n$.

SYZ Mirror Symmetry

Conjecture (Strominger, Yau, and Zaslow 1996)

When two spaces X and $\check{X}$ have dual Lagrangian almost torus fibrations over common base Q, their symplectic and complex geometry are interchanged.



Lagrangian submanifolds of $X \leftrightarrow$ Complex Submanifolds $\check{X}$

Context: HMS

More precisely, homological mirror symmetry (Kontsevich 1995) predicts the equivalence of the following categories:

- The Fukaya category of X, whose:
  - Objects are Lagrangian submanifolds and
  - $\hom(L, K)$ is the Lagrangian intersection Floer homology
- Derived Category of Coherent sheaves on $\check{X}$. Note that every complex submanifold gives a sheaf by direct image.

Naïve interchanging

Let $\underline{V} \subset Q$ be an affine submanifold so that

$$T_{\mathbb{Z}}Q \cap T\underline{V}$$

is a full lattice.

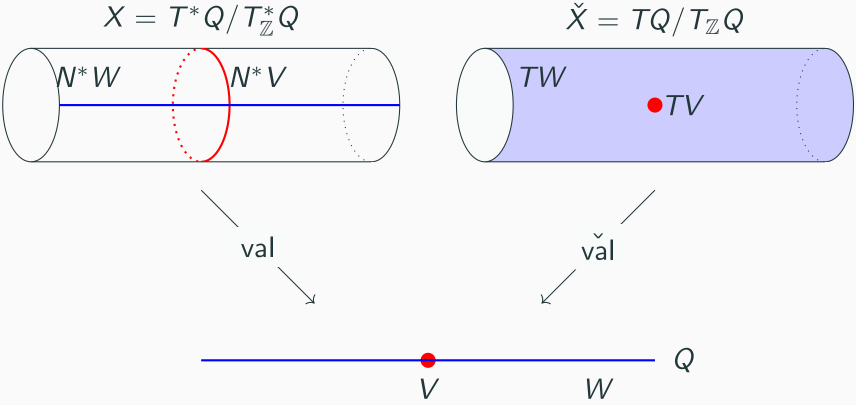
- Conormal subbundle gives us a Lagrangian submanifold

$$N^*\underline{V}/N^*_{\mathbb{Z}}\underline{V} \subset X$$

- Tangent subbundle to V gives us an almost complex submanifold

$$T\underline{V}/T_{\mathbb{Z}}\underline{V} \subset \check{X}.$$

Symplectic - Tropical - Complex Correspondence on $\mathbb{C}^*$



Background: Complex to Tropical Correspondence

Background: Tropical Curves

Definition

A *unweighted tropical curve* is a graph $V \subset Q$ whose

- Edges $V^{(0)} := \{\underline{V}_1, \ldots \underline{V}_k\}$ are all 1-dimensional affine subspaces
- At each vertex $v \in V^{(1)}$, we satisfy a balancing condition

$$\sum_{\underline{V}_i \ni v} \vec{p}_i = 0$$

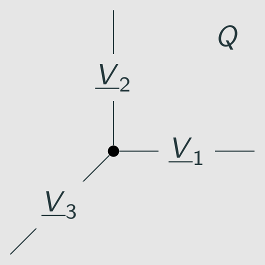
where $\vec{p}_i$ is the primitive lattice vector for $\underline{V}_k$.

There is an analogous definition for higher dimensional tropical subvarieties.

Example: Tropical Pair of Pants

Consider the polyhedral domains in $Q = \mathbb{R}^2$

$$\underline{V}_1 = \{(t,0) \;:\; t \in \mathbb{R}_{\geq 0}\}$$
$$\underline{V}_2 = \{(0,t) \;:\; t \in \mathbb{R}_{\geq 0}\}$$
$$\underline{V}_3 = \{(-t,-t) \;:\; t \in \mathbb{R}_{\geq 0}\}$$



We call this and its embeddings into $\mathbb{R}^n$ the *tropical pair of pants*.

Pair of pants, II

We consider a non-Archimedean valuation ring Λ with valuation $\text{val} : \Lambda^\times \to \mathbb{R} \cup \{\infty\}$. This gives a map $\text{trop}_B : \check{X}^\Lambda := (\Lambda^*)^2 \to Q = \mathbb{R}^2$.

Example

Consider the subscheme $\check{Y}_{pants} \subset (\Lambda^\times)^2$ given by

$$\check{Y}_{pants} = \{(\check{z}_1, \check{z}_2) \ : \ 1 + \check{z}_1 + \check{z}_2 = 0.\}$$

$$\text{val}(1 + \check{z}_1 + \check{z}_2) \geq \min(\text{val}(1), \text{val}(\check{z}_1), \text{val}(\check{z}_2))$$

$$\text{val}(\check{z}_1) = \text{val}(\check{z}_2) \qquad \text{val}(\check{z}_1) = \text{val}(1) \qquad \text{val}(\check{z}_2) = \text{val}(1)$$

B-tropicalization

Theorem (Groves and Bieri 1984)

Let $\check{Y} \subset \check{X}^{\Lambda}$ be a k-dimensional subvariety. Then $trop_B(\check{Y})$ is a k-dimensional tropical subvariety.

For us, we only consider curves in $\check{X}^{\Lambda}$, whose tropicalizations are tropical curves.

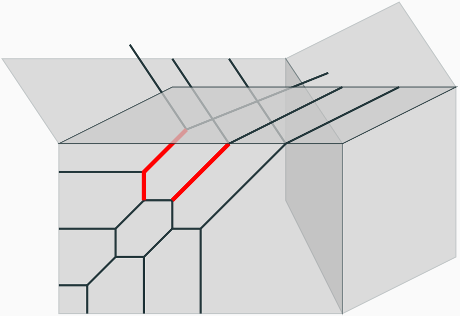
B-realization problem

Question

When does $C \in Q$ arise as the tropicalization of $\check{Y} \subset \check{X}^{\Lambda}$?

- Nishinou, Siebert, et al. 2006 If $C \subset Q$ is a tropical curve of genus 0, then C is realizable.
- Speyer 2007 If $C \subset Q$ is a tropical curve of genus 1, and is *well-spaced*, then C is realizable.

B-realization: a non-realizable curve



The cubic curve in $\mathbb{CP}^3$ has the property that it always lies inside of a plane. Therefore, this tropical curve is not realizable.

Geometric A-realization

Geometric Lagrangian Lift

Definition

A family of graded and spin Lagrangian submanifolds L_V^ϵ is a geometric Lagrangian lift of an unweighted tropical subvariety $V \subset Q$ if the following conditions hold:

- The Lagrangians L_V^ϵ are all Hamiltonian isotopic
- Over the affine portion of V,

$$L_V^\epsilon \setminus B_\epsilon(V^{(1)}) = (T^*V^{(0)}/T^*_{\mathbb{Z}}V^{(0)}) \setminus B_\epsilon(V^{(1)}).$$

In particular, $\text{syz}(L_V^\epsilon) \approx V$.

Existence of Geometric Lagrangian Lifts

Theorem (Matessi 2018; Mikhalkin 2019; Hicks 2020; Mak and Ruddat 2020)

Let $V \subset Q$ be a tropical curve. Then there exists a geometric Lagrangian lift L_V^ϵ of V.

Remark

We know how to build this lift for a large set of example of tropical varieties $V \subset Q$; for instance anything locally modelled on a hypersurface.

Is this A-realizability?

Notice that *every* tropical curve admits a geometric Lagrangian lift, while only a subset of them are B-realizable, suggesting that this is not the correct definition of "A-realizability".

Definition

We say that $V \subset Q$ is A-realizable if there exists a geometric Lagrangian lift L_V^ϵ, and furthermore L_V^ϵ is an *unobstructed Lagrangian* submanifold.

Unobstructedness is a criteria on the cancellation of holomorphic disks with boundary on L_V^ϵ.

Examples of A-realizable curves

Example

Suppose that V is a tropical curve with a single vertex (at the origin). Then L_V^ϵ is unobstructed (monotone, and bounds no holomorphic disks).

Example

Suppose that V is a tropical curve in $Q = \mathbb{R}^2$. Then for generic J, L_V^ϵ bounds no holomorphic disks and is therefore unobstructed.

Examples of A-realizable tropical subvarieties

Example

Suppose that V is a compact genus zero tropical curve in Q, the SYZ base for mirror quintic-threefold (considered in Mak and Ruddat 2020). Then L_V^ϵ is a homology sphere, and therefore unobstructed by a criteria from Fukaya 2002.

Example

Suppose that $V \subset \mathbb{R}^n$ is a hypersurface. Then L_V^ϵ is unobstructed (Hicks 2020).

So, there are many examples of where we know that L_V^ϵ is unobstructed.

A-realizability vs B-realizability

Summary of Known Results

V	A-model	B-model
Curves in $\mathbb{R}^2$	Hicks 2020	Mikhalkin 2005
Compact Genus 0 in $\dim(Q) = 3$	Mak and Ruddat 2020	Nishinou, Siebert, et al. 2006
Hypersurfaces of $\mathbb{R}^n$	Hicks 2020	Folklore
Genus 0 curves in $\mathbb{R}^n$	In Progress	Nishinou, Siebert, et al. 2006
Well-Spaced Genus 0	?	Speyer 2007

A-realizability to B-realizability

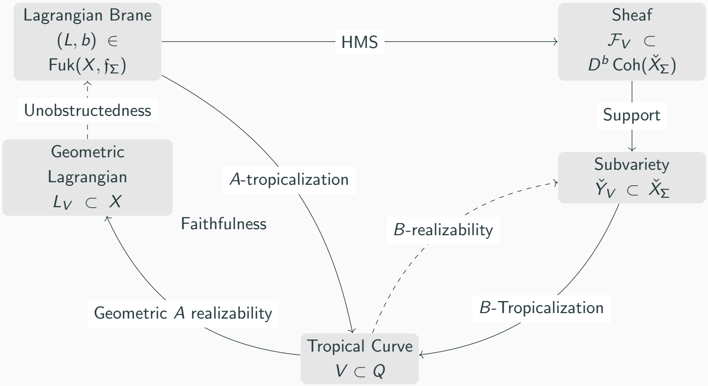
Theorem (In Progress)

Suppose that V is A-realizable. Then V is B-realizable.

Example of an application of mirror symmetry from symplectic geometry to complex geometry.

Since many V have geometric Lagrangian lifts, this is really a statement about unobstructedness of that lift.

Outline of Proof



Definition

Let $(L, b) \in \mathrm{Fuk}(X)$ be an admissible Lagrangian submanifold. The Floer theoretic support is

$$\mathrm{Supp}(L, b) := \{(F_q, \nabla) \ : \ HF^\bullet((L, b), (F_q, \nabla)) \neq 0\}$$

The A-tropicalization is

$$\mathrm{trop}_A(L, b) := \{q \mid HF^\bullet((L, b), (F_q, \nabla)) \neq 0\}.$$

This is computing the points on the mirror space contained in the support of the mirror sheaf.

Lemma (Support Lemma)

Let (L^ϵ_V, b) be the A-realization of V. Then

$$trop_A(L^\epsilon_V, b) = V.$$

Lemma (HMS Lemma)

Let $(L, b) \in \mathrm{Fuk}(X)$ be an admissible Lagrangian submanifold. Let $\mathcal{F}_L$ be the mirror sheaf to L. Then

$$trop_A(L, b) = trop_B(\mathrm{Supp}(\mathcal{F}_L)).$$

Support Lemma: Support contained within V

Because $\text{trop}_A(L^\epsilon_V, b) \subset \text{syz}(L^\epsilon_V)$ and $\text{syz}(L^\epsilon)$ approximates V, we obtain

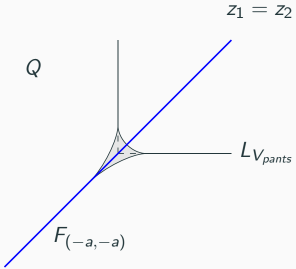
$$\text{trop}_A(L^\epsilon_V, b) \subset V.$$

Support Lemma: V contained within support

For the reverse direction, we note that whenever $q \in V^{(0)}$, the Lagrangians L_V^ϵ and F_q intersect cleanly along a subtorus of F_q. So the Lagrangian intersection Floer cohomology is a deformation of $C^\bullet(T^k)$ (where $k = n - \dim(V)$).

Idea: Given any such deformation, we can find a local system on F_q exactly cancelling out the contribution to the differential from that deformation.

Example: Pair of Pants



$$m^1(e) = T^{\omega(u^+_{e<x})} - T^{\omega(u^-_{e<x})} + T^{\omega(u_{qv})} \cdot x.$$

With local system ∇ on $F_{(-a,-a)}$

$$\begin{aligned}\langle m^1_{\nabla_{a,a}}(e), x\rangle &= \overbrace{\left(P^{\nabla_{b_1,b_2}}_{\partial u^+_{e<x}} \cdot T^{\omega(u^+_{e<x})} - P^{\nabla_{b_1,b_2}}_{\partial u^-_{e<x}} \cdot T^{\omega(\partial u^-_{e<x})}\right)}^{\text{Small Strips near } q} + \overbrace{P^{\nabla}_{u^{qv}} \cdot T^{\omega(u^{qv})}}^{\text{Large Strips}} \\ &= \exp\left(\frac{1}{2}(b_1 - b_2)\right) - \exp\left(\frac{1}{2}(b_2 - b_1)\right) + \exp\left(\frac{1}{2}(b_1 + b_2)\right) \cdot T^{-a} \\ &= T^{-a}\exp\left(-\frac{1}{2}(-b_1 - b_2)\right)\left(T^a \exp(b_1) - T^a \exp(b_2) + 1\right) \\ &= T^{-a}\exp\left(-\frac{1}{2}(-b_1 - b_2)\right)(\check{z}_1 - \check{z}_2 + 1)\end{aligned}$$

$$\mathrm{Supp}(L^{\epsilon}_{V_{pants}}) = \{(\check{z}_1, \check{z}_2) \ : \ \check{z}_1 - \check{z}_2 + 1 = 0.\}$$

Applications

Unobstructedness $\Rightarrow$ Realizability

Question

Do symplectic geometric interpretations of unobstructedness give us new insights into realizability?

V	A-model	B-model
Curves in $\mathbb{R}^2$	Hicks 2020	Mikhalkin 2005
Compact Genus 0 in $\dim(Q) = 3$	Mak and Ruddat 2020	Nishinou, Siebert, et al. 2006
Hypersurfaces of $\mathbb{R}^n$	Hicks 2020	Folklore
Genus 0 curves in $\mathbb{R}^n$	In Progress	Nishinou, Siebert, et al. 2006
Well-Spaced Genus 0	?	Speyer 2007

A symplectic proof of Genus-Zero Realizability

Lemma

Suppose V is genus zero. Then L^ϵ_V is unobstructed.

Idea of Proof for $V \in \mathbb{R}^3$.

The m^0 term for $CF^\bullet(L^\epsilon_V)$ counts holomorphic disks with boundary on L^ϵ_V.

- In this dimension, the boundary of such disks are a class in $H_1(L^\epsilon_V)$.
- Using $\chi(V) = 0$, we can show that $H_1(\partial L^\epsilon_V)$ generates $H_1(\partial L^\epsilon_V)$.
- This implies there exists a non-compact 2-chain $\mathfrak{pd}(b)$ with $\partial\mathfrak{pd}(b) = \mathfrak{pd}(m^0)$.

□

Next Steps

We've framed B-realizability in terms of the study of open Gromov-Witten invariant on the A model.

- Are there combinatorial methods to understand the OGW-invariants of tropical Lagrangian submanifolds?
- If so, do we recover new realizability criteria on the B-side?

A speculative Example

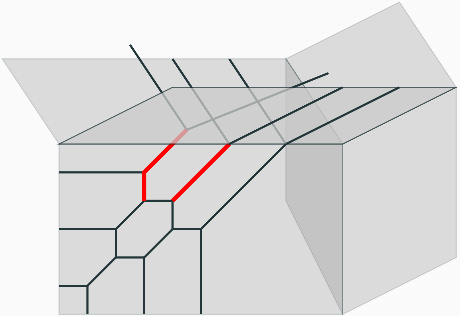
Theorem (Speyer 2007)

If the minimal distance from a cycle of a tropical elliptic curve to the non-planarity locus occurs at least twice, then the tropical curve is realizable.

Idea

If the minimal area of holomorphic disks with boundary on L_V^ϵ occurs at least twice, you stand a decent chance of being unobstructed.

Potential Application: Speyer's Realizability



References

Fukaya, Kenji (2002). "Floer homology for families-a progress report". *CONTEMPORARY MATHEMATICS* 309, pp. 33–68.

Groves, JRJ and Robert Bieri (1984). "The geometry of the set of characters iduced by valuations.".

Hicks, Jeffrey (2020). "Tropical Lagrangian hypersurfaces are unobstructed". *Journal of Topology* 13.4, pp. 1409–1454.

Kontsevich, Maxim (1995). "Homological algebra of mirror symmetry". *Proceedings of the international congress of mathematicians*. Springer, pp. 120–139.

Mak, Cheuk Yu and Helge Ruddat (2020). "Tropically constructed Lagrangians in mirror quintic threefolds". *Forum of Mathematics, Sigma*. Vol. 8. Cambridge University Press.

Matessi, Diego (2018). "Lagrangian pairs of pants". *International Mathematics Research Notices*.

Mikhalkin, Grigory (2005). "Enumerative tropical algebraic geometry in $\mathbb{R}^2$". *Journal of the American Mathematical Society* 18.2, pp. 313–377.

— (2019). "Examples of tropical-to-Lagrangian correspondence". *European Journal of Mathematics* 5.3, pp. 1033–1066.

Nishinou, Takeo, Bernd Siebert, et al. (2006). "Toric degenerations of toric varieties and tropical curves". *Duke Mathematical Journal* 135.1, pp. 1–51.

Speyer, David E (2007). "Uniformizing tropical curves I: genus zero and one". *arXiv preprint arXiv:0711.2677*.

Strominger, Andrew, Shing-Tung Yau, and Eric Zaslow (1996). "Mirror symmetry is T-duality". *Nuclear Physics B* 479.1-2, pp. 243–259.