Floer Theory and Lagrangian Cobordisms

Jeff Hicks

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University of Edinburgh

Summary

Introduction

Applications of Lagrangian cobordisms

Decompositions and unobstructedness of Lagrangian cobordisms

Floer cohomology of surgery traces

Introduction

Motivation

How do we realize algebraic relations in the Fukaya category in terms of geometric relations between Lagrangian submanifolds?

Exact isotopy/homotopy

Geometric Relation:

Definition

Let $i_t : L \to X$ be a Lagrangian isotopy (homotopy), so that $i_t^*(\omega) = 0$. We say that this is an exact isotopy (homotopy) if $i_t^*\omega\left(\frac{di_t}{dt}, -\right)$ is exact for all t.

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If L^-, L^+ are exactly isotopic, then they are isomorphic objects in $\text{Fuk}(X)$.

Sometimes this is true if L^-, L^+ are exactly homotopic, but need to consider bounding cochains.

Lagrangian Surgeries

Geometric Relation:

Polterovich connect sum of Lagrangians L^1, L^2

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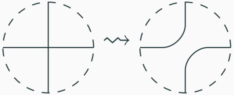
$L^1 \cup L^2 =$  $= L^1 \# L^2$

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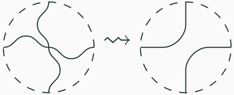
$L^+ =$  $= L^-$

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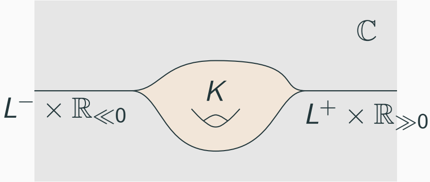
If L^+ is immersed, we can sometimes give it a bounding cochain to make it isomorphic to L^-.

Geometric relation: Lagrangian Cobordisms

Definition (Arnol'd)

Let L^- and L^+ be Lagrangian submanifolds of X. A two ended Lagrangian cobordism $K : L^+ \rightsquigarrow L^-$ is a Lagrangian in $X \times \mathbb{C}$ with ends limiting to $L^- \times \mathbb{R}_{\ll 0}$ and $L^+ \times \mathbb{R}_{\gg 0}$.



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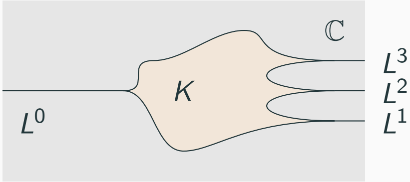
There is also a definition for cobordisms with multiple ends $K : (L^1, \ldots, L^k) \rightsquigarrow L^0$.

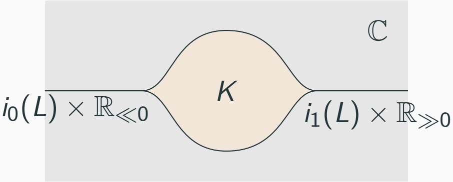


Example I: Suspension of Exact Homotopy

Let $i_t : L \to X$ be an exact Lagrangian homotopy with primitive $H_t : L \to \mathbb{R}$ satisfying $dH_t = i_t^*\omega\left(\frac{di_t}{dt}, -\right)$, $t \in [0,1]$. The suspension cobordism

$$\begin{aligned} L \times \mathbb{R} &\hookrightarrow X \times \mathbb{C} \\ (x,t) &\mapsto (i_t(x), t + \sqrt{-1}H_t(x)) \end{aligned}$$

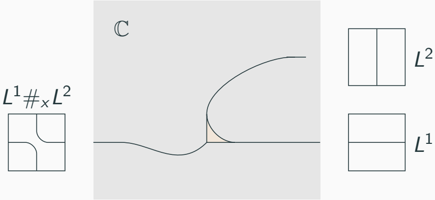
is a Lagrangian cobordism between $i_0(L)$ and $i_1(L)$.



Example II: Connect Sum and Cobordism

Let L^1, L^2 be Lagrangian submanifolds with transverse intersection at a point x . There is a *surgery trace cobordism*

$$K : (L^1, L^2) \rightsquigarrow L^1 \#_x L^2.$$



Algebraic Relations from Cobordisms

Work of Biran and Cornea; Nadler and Tanaka show that *monotone* Lagrangian cobordisms $K : L \rightsquigarrow (L^1, \ldots L^k)$ give iterated exact sequences

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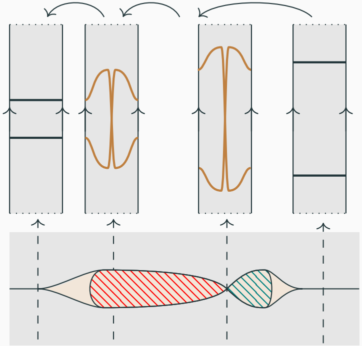
Suppose that embedded Lagrangian submanifolds L^-, L^+ are Lagrangian homotopic. Then there exists an embedded Lagrangian cobordism (possibly non-oriented) between L^-, L^+.



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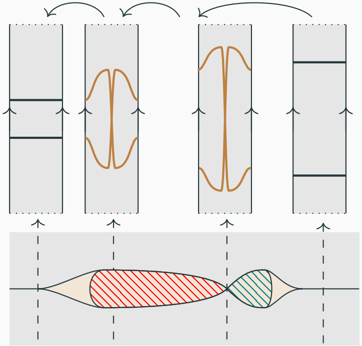

We should only consider Lagrangians which are unobstructed (by bounding cochain).

Applications of Lagrangian cobordisms

Two Applications

Using Lagrangian cobordisms...

- ... to produce “nice” resolutions in the Fukaya category
- ... to identify isomorphic Lagrangian branes

Constructing Exact Lagrangian cobordisms

Proposition (Hanlon-H)

Let (X, λ) be a Liouville domain, and L be an exact Lagrangian submanifold with primitive $f : L \to \mathbb{R}$. There exists an exact Lagrangian cobordism $K : L \rightsquigarrow (L^1, \ldots, L^k)$, where each L^i is a disjoint union of cocores.

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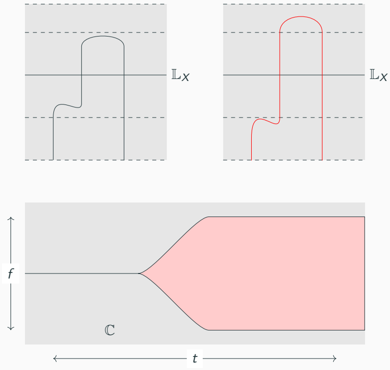

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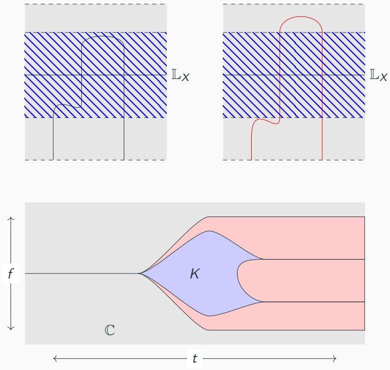

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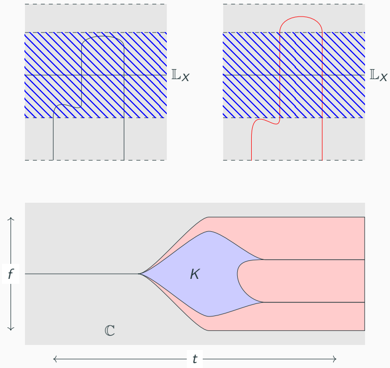
Furthermore, let $t_1 < \ldots < t_k$ be the values of $f(L \cap \mathbb{L}_X)$. Then

$$L^i = \bigcup_{x \in f^{-1}(t_i)} \text{cocore}(x).$$



Constructing Exact Lagrangian cobordisms

Given $N \subset M$ a submanifold, use $f : N \to \mathbb{R}$ a Morse function to define a perturbation of the conormal bundle $N^*_f N \subset T^*M$.

$$\lambda|_{N^*_f N} = \pi^* df$$
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Corollary

Let $N \subset M$ be a manifold. We can express $N^* N$ as

$$\left[\left(\bigoplus_{x \in f^{-1}(t_k)} T^*_x M\right) \to \cdots \to \left(\bigoplus_{x \in f^{-1}(t_1)} T^*_x M\right)\right]$$

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A similar statement appears in [GPS18].

Leveraging geometric generation

Definition

Let $\mathcal{C}$ be a category split-generated by $G \in \mathcal{C}$.

- The split-generation time $\text{⏲}_G(L)$ is the minimum number of mapping cones needed to express $L \in \mathcal{C}$ in terms of sums, summands, and shifts of $G \in \mathcal{G}$.
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The bound $\text{Rdim}(\mathcal{W}(T^*M)) \leq \dim(M)$ was observed by Bai and Côté

HMS Application

A similar result can be proven about partially wrapped Fukaya categories of cotangent bundles. By applying homological mirror symmetry for toric varieties:

Corollary

Let X_Σ be a smooth projective toric variety. Then the Rouquier dimension of X_Σ is $\dim_{\mathbb{C}}(X_\Sigma)$.

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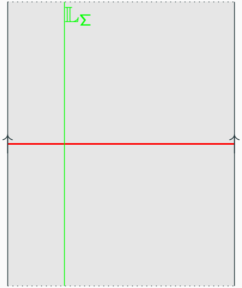

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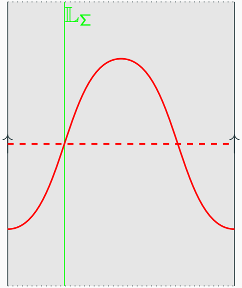

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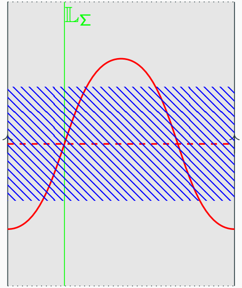

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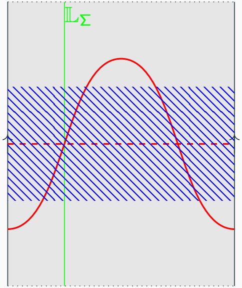

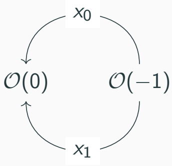

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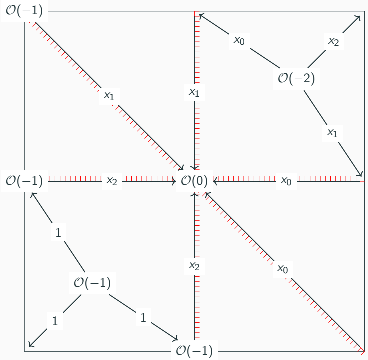
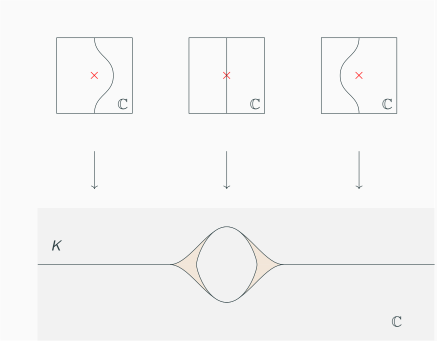
Example: Resolving $\mathcal{O}_{pt}$ in $\mathbb{CP}^2$



Example: Concatenating surgery with anti-surgery



Consider the Lefschetz fibration

$$W : \mathbb{C}^2 \xrightarrow{xy} \mathbb{C}.$$

Consider the concatenation of the surgery trace cobordisms for two thimbles.

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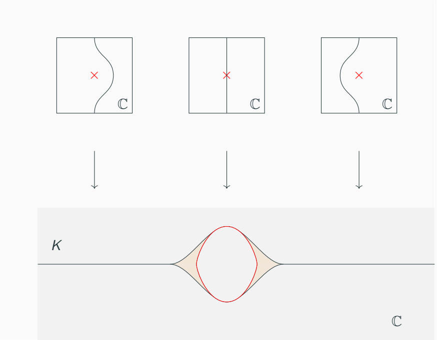

The red loop is the boundary of a regular Maslov 0 holomorphic disk u. u (and its multiple covers) are the only disks with boundary on K for the standard choice of almost complex structure.

We can find a non-compact 2-chain $b_0 \subset K$ so that

$$\partial u = \partial b_0.$$

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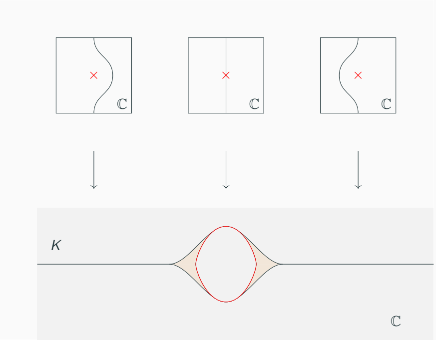

It follows that K has a bounding cochain b whose lowest valuation term is $T^{\omega(u)}b_0$. If a multiple cover formula holds we can compute the higher order terms.

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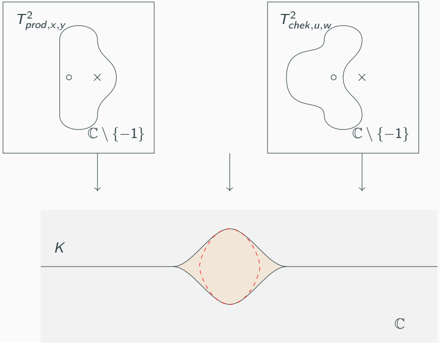

We use this model to obtain an *unobstructed* Lagrangian cobordism between monotone product and Chekanov tori,

$$T^2_{prod,x,y}, T^2_{chek,u,w} \subset \mathbb{C}^2 \setminus \{-xy = 1\}$$

$$(K, b) : (T^2_{chek,u,w}, b_{chek,u,w}) \rightsquigarrow (T^2_{prod,x,y}, b_{prod,x,y}).$$

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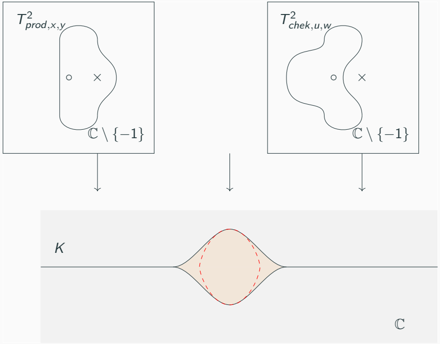

$$\mathcal{MC}(K) \xrightarrow{\pi_*^-} \mathcal{MC}(T^2_{chek,u,w}), \qquad \mathcal{MC}(K) \xrightarrow{\pi_*^+} \mathcal{MC}(T^2_{prod,x,y})$$

By interpreting the bounding cochains on ends as a local systems we identify flux charts on Chek/Prod tori by

$$(u, w) \mapsto (u/(w-1), uw/(w-1)).$$

Matches Auroux wall-crossing formula.

Decompositions and unobstructedness of Lagrangian cobordisms

Motivation

Want to understand...

- Under what conditions are cobordant Lagrangians $K : L^+ \rightsquigarrow L^-$ equivalent in the Fukaya category?
- Under these conditions, can we recover the map $CF^\bullet(L^+) \to CF^\bullet(L^-)$ from the geometry/topology of K?

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Strategy: Smooth cobordisms can be decomposed into standard pieces. We will decompose K into standard pieces, and analyze each piece.

Lagrangian $(k, n-k+1)$ surgery trace (Audin, Lalonde, and Polterovich)

For $0 \leq k \leq n$, the local Lagrangian $(k, n-k+1)$ surgery trace is the Lagrangian submanifold $K^{k,n-k+1} \subset (\mathbb{C})^n \times \mathbb{C}$ parameterized by $j^{k,n-k+1} : \mathbb{R}^{n+1} \to \mathbb{C}^n \times \mathbb{C}$

$$(x_0, x_1, \ldots, x_n) \mapsto \left(x_1 + \sqrt{-1}\sigma_{1,k} 2x_1 x_0, \ldots, x_n + \sqrt{-1}\sigma_{n,k} 2x_n x_0, x_0^2 + \sum_{i=1}^n \sigma_{i,k} x_i^2 - \sqrt{-1}x_0\right).$$

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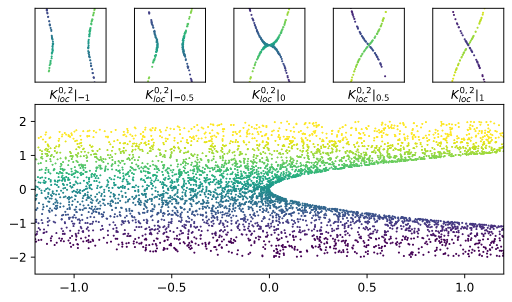
In the top row: the surgery of 2 curves.

In the bottom row: the Lagrangian surgery trace projected to the cobordism coordinate.



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We say that $K : L^+ \rightsquigarrow L^-$ is a $(k, n-k+1)$ surgery trace if there is a small neighborhood U so that $L^+ \cap U, L^- \cap U$ are the ends of the local surgery trace.

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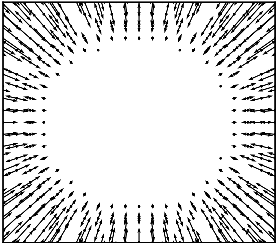
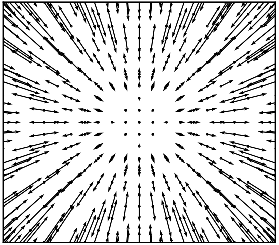
Two Lagrangian submanifolds in $T^*\mathbb{R}^2$, drawn as a collection of covectors, differing by Lagrangian $(0,3)$ surgery (Polterovich surgery)



$K^{0,3}_{loc}|_{-0.5}$



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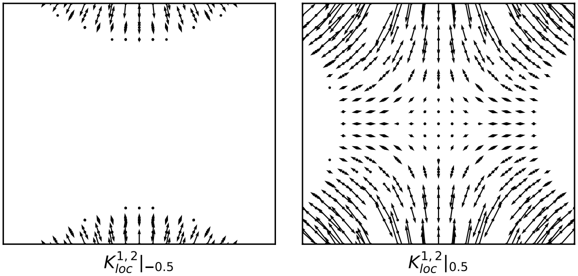
Two Lagrangian submanifolds in $T^*\mathbb{R}^2$, drawn as a collection of covectors, differing by Lagrangian $(1, 2)$ surgery. The surgery collapses an isotropic immersed S^1.



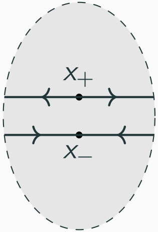
Floer cochains of the surgery trace

When $K^{k,n-k+1} : L^+ \rightsquigarrow L^-$ is a graded Lagrangian surgery trace cobordism, there exist Morse functions

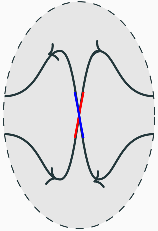
$$f^{\pm} : L^{\pm} \to \mathbb{R}$$

agreeing away from the surgery neighborhood such that inside the surgery neighborhood, the critical points and self-intersections have the following degrees

Index	Self-Inter. of L^+	Crit. pts. of f^+	Crit. pts. of f^-
$k+1$	$(q_+ \to q_-)$		x_+
$n-k-1$	$(q_- \to q_+)$		x_-



L^-



L^+

Decompositions of Lagrangian cobordisms

Theorem (H)

Let $K : L^+ \rightsquigarrow L^-$ be a Lagrangian cobordism. K is exactly homotopic to the concatenation of surgery trace cobordisms and suspensions of exact homotopies.

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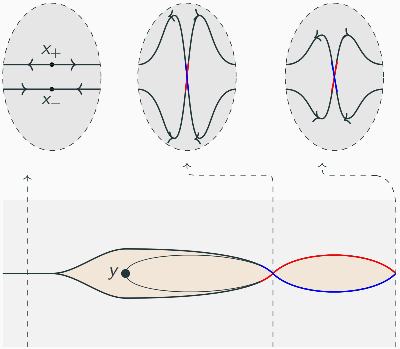
Does this extend to homotopy equivalence?

Floer cohomology of surgery traces

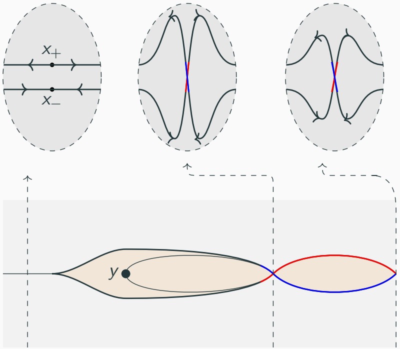
Floer cochains of immersed surgery trace

In order to discuss the Floer cohomology for Lagrangian cobordisms with immersed ends, we need to apply a small perturbation to the ends of the Lagrangian cobordisms so that they have transverse intersections (bottleneck trick, Mak and Wu).



Floer cochains of immersed surgery trace

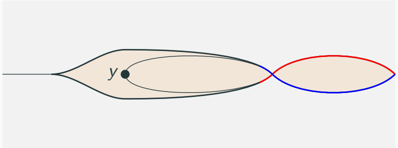
We take a Morse function for the surgery trace which agrees with $\mathrm{Re}(z)$ away from the ends of the Lagrangian cobordism, and agrees with $f^{\pm}$ over the ends.



Floer cochains of immersed surgery trace

We obtain the following Floer cochains for K

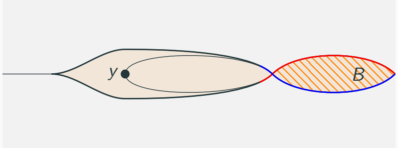
Index				
$k+1$	x_+			$(q_+, 1) \to (q_-, 1)$
$k+2$			$(q_+, 0) \to (q_-, 0)$	
$n-k-1$	x_-		$(q_-, 0) \to (q_+, 0)$	
$n-k$		y		



Floer cochains of immersed surgery trace

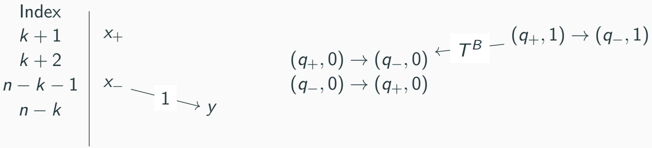
There is a strip of area B pairing the left and right self-intersections.

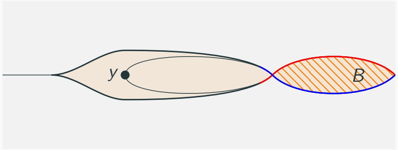
Index			
$k+1$	x_+		$\leftarrow T^B - (q_+,1) \to (q_-,1)$
$k+2$			$(q_+,0) \to (q_-,0)$
$n-k-1$	x_-		$(q_-,0) \to (q_+,0)$
$n-k$		y	



Floer cochains of immersed surgery trace

There is an isolated Morse flow line from x_- and y.





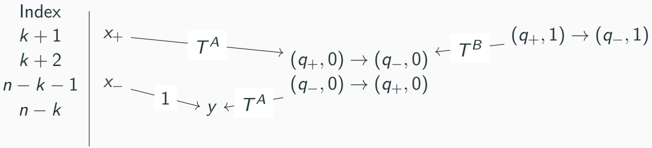
Floer cochains of immersed surgery trace

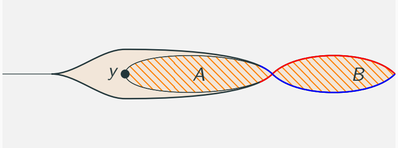
Theorem (H)

There exist Floer trajectories from

$$x_+ \text{ to } ((q_+, 0) \to (q_-, 0))$$
$$((q_-, 0) \to (q_+, 0)) \text{ to } y$$

arising from a holomorphic teardrop of area A.





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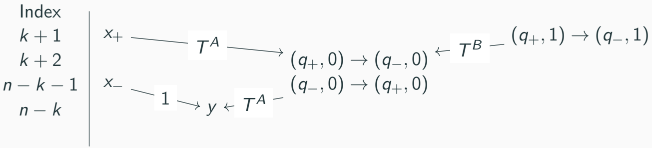
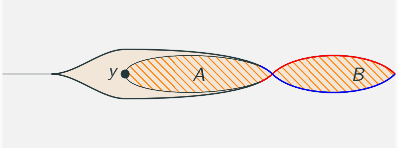
This suggests that chains on LHS and RHS are cohomologically identified. (Warning: valuations!)

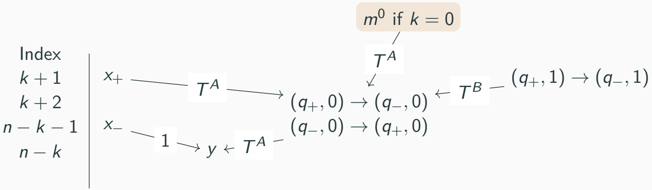


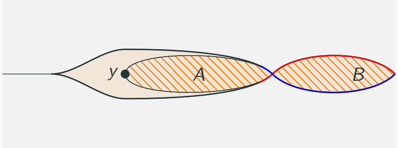


Floer cochains of immersed surgery trace

When $k = 0$ (setting of Polterovich surgery), the teardrop is isolated, contributing to an m^0 term

$$T^A \cdot ((q_+, 0) \to (q_-, 0)).$$





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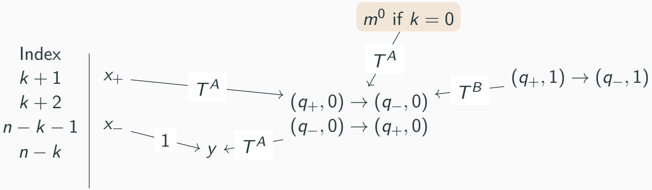
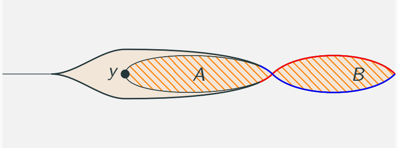
When $B < A$,

$$T^{A-B} \cdot ((q_+, 1) \to (q_-, 1))$$

is a candidate bounding cochain.





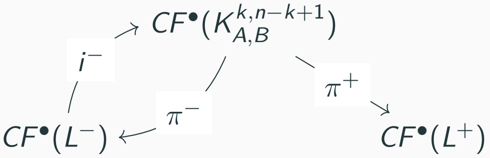
Speculation

We expect that there are filtered A_∞ projections from the Floer A_∞ algebra of a Lagrangian cobordism to those of the ends.

$$CF^\bullet(L^-) \xleftarrow{\pi^-} CF^\bullet(K_{A,B}^{k,n-k+1}) \xrightarrow{\pi^+} CF^\bullet(L^+)$$

Speculation

Previous discussion suggests: there exists a *weakly filtered* map of curved A_∞ algebras $i^- : CF^\bullet(L^-) \to CF^\bullet(K^{k,n-k+1})$ so that $i^- \circ \pi^-$ is *weakly filtered homotopic* to the identity.



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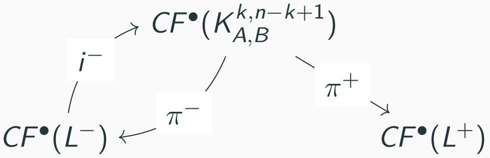
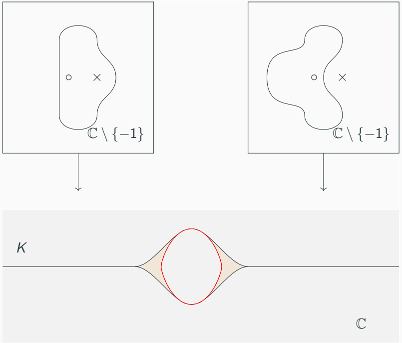


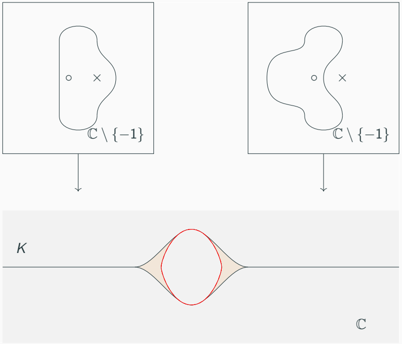
Speculation: return to Chekanov and Product tori

Bounding cochain: Recall that we had a Lagrangian cobordism between the Chekanov and Product type tori in $\mathbb{C}^2 \setminus \{xy = -1\}$. The ends of this Lagrangian cobordism, when equipped with appropriate local systems, are isomorphic objects in the Fukaya category



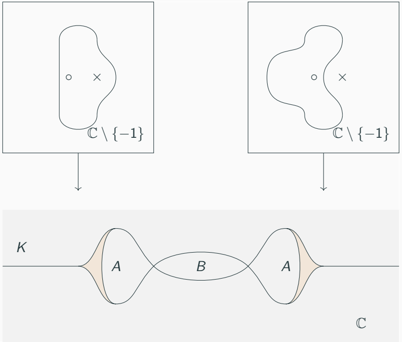
Speculation: return to Chekanov and Product tori

$$\begin{array}{ccc} & \mathcal{MC}(K) & \\ \swarrow^{\pi_*^-} & & \searrow^{\pi_*^+} \\ \mathcal{MC}(T^2_{chek,u,w}) & & \mathcal{MC}(T^2_{prod,x,y}) \end{array}$$



Speculation: return to Chekanov and Product tori

Bounding cochain: We apply the cobordism decomposition result, giving an immersed Lagrangian cobordism which is the concatenation of a surgery, exact homotopy, and anti-surgery. Since $B < A$, we can find a bounding cochain cancelling the contribution from the holomorphic teardrop.

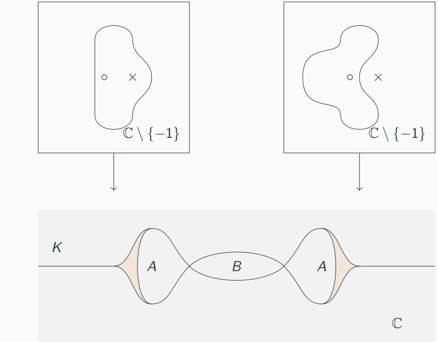


Speculation: return to Chekanov and Product tori

$$\mathcal{MC}(T^2_{chek,u,w}) \xleftarrow{\pi_*^-} \mathcal{MC}(K_1) \xrightarrow{\pi_*^+} \mathcal{MC}(S^2_{whit}) \xleftarrow{\pi_*^-} \mathcal{MC}(K_2) \xrightarrow{\pi_*^+} \mathcal{MC}(T^2_{prod,x,y})$$

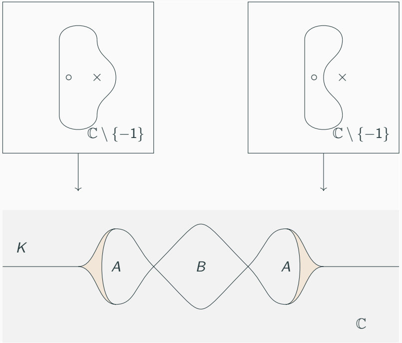
There exists bounding cochains $b_1 \in \mathcal{MC}(K_1), b_2 \in \mathcal{MC}(K_2)$ so that

$$\pi_*^+(b_1) = \pi_*^-(b_2).$$



Speculation: return to Chekanov and Product tori

No Bounding cochain: If we sweep out more flux over the exact homotopy, the ends of the Lagrangian cobordism can be disjoined (and in particular are not isomorphic!). In this setting, $B \geq A$, and it is not possible to find a bounding cochain cancelling the m^0 term arising from the count of teardrops.



Speculation: return to Chekanov and Product tori

$$\mathcal{MC}(T^2_{chek,u',w'}) \xleftarrow{\pi_*^-} \mathcal{MC}(K_1) \xrightarrow{\pi_*^+} \mathcal{MC}(S^2_{whit}) \xleftarrow{\pi_*^-} \mathcal{MC}(K_2) \xrightarrow{\pi_*^+} \mathcal{MC}(T^2_{prod,x,y})$$

There are no bounding cochains $b_1 \in \mathcal{MC}(K_1), b_2 \in \mathcal{MC}(K_2)$ so that

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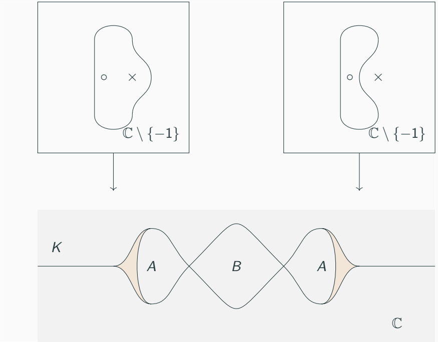

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