Wall Crossings from Lagrangian Cobordisms

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Summary

Introduction

Floer Theory and disk counts

Our geometric setup:

- X a symplectic manifold with compatible almost complex structure,
- $L \subset X$ be a Lagrangian submanifold
- $h : L \to \mathbb{R}$ a Morse function

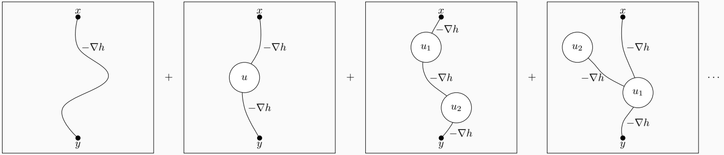
Piece 1: $CF^\bullet(L, h)$

Let L be a Lagrangian submanifold. Pick $h : L \to \mathbb{R}$ a Morse function.

Definition (Fukaya and Oh 1997; Biran and Cornea 2008; Charest and Woodward 2017)

The *Pearly-Floer* algebra is a filtered A_∞ algebra where

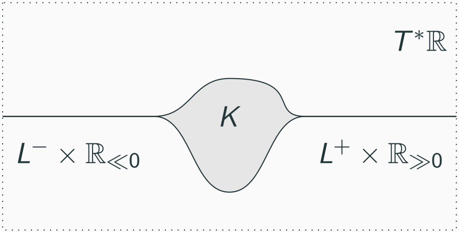
- $CF^\bullet(L, h) := \Lambda\langle \mathrm{crit}(h)\rangle$
- Maps $m^k : CF^\bullet(L, h)^{\otimes k} \to CF^\bullet(L, h)$ given by counting configurations of flow trees with holomorphic disk insertions



Piece 2: Lagrangian Cobordisms

Definition

Let L^- and L^+ be Lagrangian submanifolds of X. A Lagrangian cobordism $K : L^- \rightsquigarrow L^+$ is a Lagrangian in $X \times T^*\mathbb{R}$ with ends limiting to $L^- \times \mathbb{R}_{\ll 0}$ and $L^+ \times \mathbb{R}_{\gg 0}$.



Piece 3: Invariance of Floer theory

When L is monotone:

- $CF^\bullet(L, h)$ is a chain complex giving the *Lagrangian quantum cohomology.*
- Given L^- and L^+ which are Hamiltonian isotopic, then there is a quasi-isomorphism $CF^\bullet(L^-, h) \cong CF^\bullet(L^+, h)$.
- More generally the work of Biran and Cornea 2013 shows that if L and $L+$ are *monotone Lagrangian cobordant*, then we expect that $CF^\bullet(L^-) \cong CF^\bullet(L^+)$.

Outline of Talk

Our goal is to weaken the monotonicity assumption and see what we can still recover.

- The algebra $CF^\bullet(L, h)$ is now an example of a filtered A_∞ algebra.
  - Define and motivate usage of A_∞ algebras.
  - 2 Properties: *deformations* and *homotopy transfer theorem*.
- Prove invariance of $CF^\bullet(L, h)$ by analogy to topology.
- Say something about the specific example of Chekanov / Product tori in $\mathbb{C}^2 \setminus \{z_1 z = 1\}$

Goal I: Construction of Continuation Maps

Theorem

Let $H : X \to \mathbb{R}$ be a Hamiltonian whose Hamiltonian flow realizes an isotopy between L^- and L^+. Then there is a continuation map between

$$\Phi_{H_t} : CF^\bullet(L^-) \to CF^\bullet(L^+).$$

We will prove this using Lagrangian cobordisms, and by analogy to topology and Morse theory.

Goal II: Construction of Continuations from Cobordisms

By extending our analogy we'll see an interesting example of a continuation map from a non-monotone Lagrangian cobordism.

Example

There exists a Lagrangian cobordism between the monotone Product and Chekanov tori $L^-_{T^2}$ and $L^+_{T^2}$ in $(\mathbb{C})^2 \setminus \{z_1 z_2 = 1\}$. Using the same framework for continuation will give us a map:

$$\Theta_K : CF^\bullet{}_{b^-}(L^+_{T^2}) \to CF^\bullet{}_{b^+}(L^-_{T^2})$$

Previous work:

- Most of these ideas are implicitly in Biran and Cornea 2013.
- Continuation maps already constructed in Fukaya, Oh, Ohta, and Ono 2010.
- Main geometry example from Auroux 2007.
- General Wall-Crossing constructions: Pascaleff and Tonkonog 2020; Rizell, Ekholm, and Tonkonog 2018; Palmer and Woodward 2019.

New ideas are a consistent packaging to bring these lines of thought together.

Expositional Choices

- We avoid technical details on the construction of $CF^\bullet(L, h)$.
- Always working over Novikov ring.
- $\pm$ signs are ignored.

Filtered A_∞ algebras

Definition of (A, m^k)

Filtered A_∞ algebras are a generalization of differential graded algebras

Definition

A *filtered A_∞ algebra* is a graded vector space A, along with a collection of maps for $k \geq 0$

$$m^k : A^{\otimes k} \to A[2-k].$$

satisfying the *curved A_∞ relations*

$$0 = \sum_{i_1+j+i_2=k} m^{i_1+1+i_2}(\mathrm{id}^{\otimes i_1} \otimes m^j \otimes \mathrm{id}^{\otimes i_2}).$$

If $m^0 = 0$, we say that A is *uncurved*.

Meaning of first few A_∞ relations

$$0 = \sum_{i_1+j+i_2=k} m^{i_1+1+i_2}(\mathrm{id}^{\otimes i_1} \otimes m^j \otimes \mathrm{id}^{\otimes i_2}).$$

k	Uncurved Relation
0	0
1	$m^1 \circ m^1 = 0$
2	$m^1 \circ m^2 = m^2(m^1 \otimes \mathrm{id}) + m^2(\mathrm{id} \otimes m^1)$

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- Note that if $m^0 = 0$, then (A, m^1) is a chain complex.
- If uncurved, then cohomology of A inherits has product from m^2.

The price of filtered A_∞

- *Additional Data:* Morphisms of A_∞ algebras are now collections of maps

$$f^k : A^{\otimes k} \to B.$$

- *Defined up to Homotopy:* Definitions of tensor products, homotopy, etc. all have choices involved.
- *Proceed with caution:* Many things are unintuitive:
  - No zero morphism!
  - Fiber Products still exist,
  - Mapping cones do not exist.

Why A_∞ algebras?

We use A_∞ algebras because they satisfy two properties that standard differential graded algebras do not.

- Their deformations are better behaved.
- They satisfy a homotopy transfer principle.

Deformations for A_∞ algebras

Definition

Let (A, m^k) be a filtered A_∞ algebra and $b \in A$ any element (of positive Novikov valuation). The b-deformed A_∞ structure is given by

$$m_b^k(a_1 \otimes \cdots \otimes a_k) = \sum_{i_0, \ldots, i_k \in \mathbb{N}} m^{i_0 + \ldots + i_k + k}(b^{\otimes i_0} \otimes a_1 \otimes b^{\otimes i_1} \otimes \cdots \otimes b^{\otimes i_{k-1}} \otimes a_k \otimes b^{\otimes i_k})$$

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Claim

(A, m_b^k) *is again filtered* A_∞ *algebra.*

Example: DGA

- Let $(C, d, \wedge)$ be a differential graded algebra.
- $a \in C$

We can then deform the differential to:

$$d_a := d + a\wedge$$

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We can then deform the differential to:

$$d_a := d + a\wedge$$

Remains a DGA if $a \wedge a = 0$ and $da = 0$. Is still a filtered A_∞ algebra regardless of what a is.

Maurer-Cartan Solutions

Definition

(A, m^k) is called *unobstructed* if there exists a deformation $b \in A$ so that

$$m_b^0 = \sum_{k=0}^{\infty} m^k(b^{\otimes k}) = 0$$

The set all such deformations is called the *Maurer Cartan space,*

$$\mathcal{MC}(A) := \{b \in A \mid m_b^0 = 0\}$$

Why do we care: (A, m_b^k) now is uncurved, and has cohomology.

Deformations and Morphisms

Claim

Given a morphism $f^k : A^{\otimes k} \to B$, there is a pushforward morphism

$$\begin{aligned} f_* : \mathcal{MC}(A) \to& \mathcal{MC}(B) \\ b \mapsto& \sum_k f^k(b^{\otimes k}). \end{aligned}$$

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Corollary

(A, m^k) *is unobstructed if and only if there exists* $f : 0 \to A$.

Limitation: Homotopy Inverses

Limitation of DGA is that homotopy equivalences $\pi : B \to A$ do not necessarily have inverses.

Example (De Rahm Theory)

$$\mathcal{H}^\bullet(X;\mathbb{R}) \underset{\pi}{\overset{i}{\leftrightarrows}} \Omega^\bullet(X;\mathbb{R}) \circlearrowleft h$$

Usually map i is not a map of algebras!

Homotopy Transfer Principle

Theorem (Homotopy Transfer Principle Kadeishvili 1980)

Let B be a uncurved A_∞ algebra, and (A, d_A) be a chain complex.

$$A \underset{\pi}{\overset{i}{\rightleftarrows}} B \circlearrowleft h$$

$$\begin{aligned} \pi \circ m_B^1 - d_A \circ \pi &= 0 \\ i \circ d_A - m_B^1 \circ i &= 0 \\ h \circ m_B^1 + m_B^1 \circ h &= \mathrm{id} - i \circ \pi \end{aligned}$$

Then we can extend the maps π and d and i to filtered A_∞ structures.

Curved Homotopy Transfer Principle

Theorem (Curved HTP)

Let B be a filtered A_∞ algebra, and (A, d_A) be a module with differential.

$$A \underset{\pi}{\overset{i}{\rightleftarrows}} B \circlearrowleft h$$

$$\begin{aligned}
\pi \circ m_B^1 - d_A \circ \pi =& \pi \circ \left(m_B^2(h, m_B^0) + m_B^2(m^0, h) \right) \\
i \circ d_A - m_B^1 \circ i =& h(m_B^2(i, m_B^0) + m_B^2(m_B^0, i)) \\
h \circ m_B^1 + m_B^1 \circ h =& \mathrm{id} - i \circ \pi + h \circ (m_B^2(m_B^0 \otimes h) + m_B^2(h \otimes m_B^0)) \\
\pi \circ h =& 0 \qquad h \circ i = 0
\end{aligned}$$

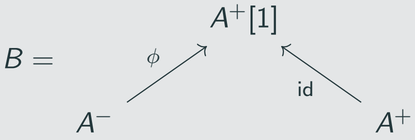
Then we can extend the maps π and d and i to A_∞ structures.

Why do we need Curved HTT?

We'll primarily be using the curved homotopy transfer theorem for the following construction:

Definition

A *mapping cylinder* from A^+ to A^- is an A_∞ algebra B whose differential decomposes as



where A^- and A^+ are A_∞ algebras.

Continuation Maps from Mapping Cocylinders

Mapping cylinders give us the setup of the homotopy transfer theorm:

$$A^- \underset{\pi^-}{\overset{i^-}{\rightleftarrows}} B \circlearrowleft h$$

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$$B \xrightarrow{\pi^+} A^+$$

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Additionally, we a have natural projection

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The composition of i and π gives us continuation map associated to the cylinder.

$$\Theta_B : A^- \to A^+.$$

Pearly Model of Lagrangian Cobordisms

Recalling Pearly Floer model: $CF^\bullet(L, h)$

- $CF^\bullet(L, h)$ is a filtered A_∞ algebra.
- $CF^\bullet(L, h)$ is a deformation of the Morse algebra $CM^\bullet(L, h)$ by the count of treed disks.

We frequently rather work in the uncurved setting.

Definition

If there exists a $b \in CF^\bullet(L, h)$ so that the b-deformed product has no curvature,

$$m_b^0 = 0$$

we say that the Lagrangian is *unobstructed.* We write ${CF^\bullet}_b(L, h)$.

Note: $m^0 = m^1(b)$ at first order.

Restriction to ends

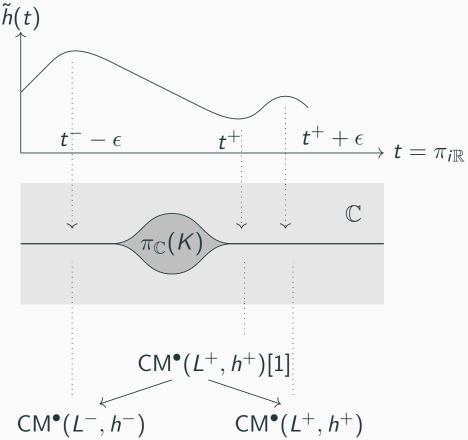
We will argue against the following analogy to Morse theory:

Claim

Let K be a cobordism. If $h : K \to \mathbb{R}$ is chosen with appropriate behaviour near $\partial K = L^- \sqcup L^+$, there exists projections on the Morse complex

$$\begin{array}{ccc} & CM^\bullet(K, h) & \\ {\swarrow}{\scriptstyle \pi^+} & & {\scriptstyle \pi^-}{\searrow} \\ CM^\bullet(L^-, h^-) & & CM^\bullet(L^+, h^+) \end{array}$$

Bottlenecked Morse Function



Restriction to ends

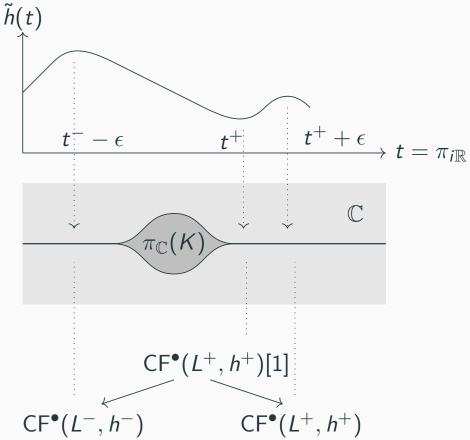
We expect that this should carry over to the Floer theoretic case.

Assumption

If $h : K \to \mathbb{R}$ is an appropriately chosen Morse function (next slide), there exists projections

$$CF^\bullet(L^-, h^-) \xleftarrow{\pi^+} CF^\bullet(K, h) \xrightarrow{\pi^-} CF^\bullet(L^+, h^+)$$

Bottlenecked Floer Theory



Continuation Maps From Cylindrical Cobordism

Example: Suspension of Hamiltonian isotopy

Let $H_t : X \times \mathbb{R}_t \to \mathbb{R}$ be a time dependent Hamiltonian supported on $t \in (0, 1)$. Let θ^t be the time t-flow and $L \subset X$ a Lagrangian. The suspension cobordism $K_{H_t} \subset X \times T^*\mathbb{R}$ parameterized by

$$\begin{aligned} L \times \mathbb{R} &\hookrightarrow X \times T^*\mathbb{R} \\ (x, t) &\mapsto (\theta^t(x), (t, H_t(x))) \end{aligned}$$

is a Lagrangian cobordism between L and $\theta^1(L)$.

Continuation Maps

Let $H_t : X \times \mathbb{R} \to \mathbb{R}$ be a Hamiltonian. With appropriate bottlenecked Morse function the suspension cobordism

$$CF^\bullet(K_{H_t}, h) = \begin{array}{ccccc} & & CF^\bullet(L^+, h^+)[1] & & \\ & \overset{\phi}{\nearrow} & & \overset{\text{id}}{\nwarrow} & \\ CF^\bullet(L^-, h^-) & & & & CF^\bullet(L^+, h^+) \end{array}$$

is an example of a filtered A_∞ mapping cylinder.

Continuation Maps

By applying the curved homotopy transfer theorem to the configurations

$$CF^\bullet(L^-, h^-) \underset{\pi^-}{\overset{i^-}{\rightleftarrows}} CF^\bullet(K_{H_t}, h) \xrightarrow{\pi_+} CF^\bullet(L^+, h^+)$$

$$\circlearrowleft h$$

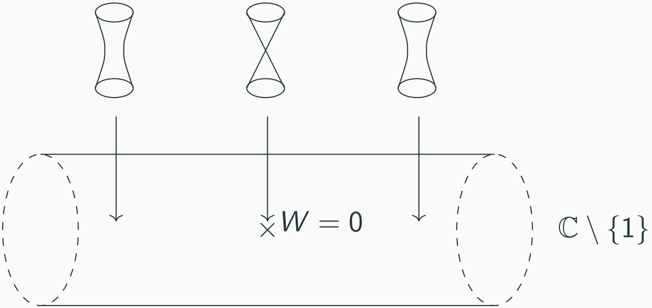
Theorem

There exists an A_∞ morphism given by the composition

$$\Theta_{H_t} := \pi^+ \circ i^- : CF^\bullet(L^-, h^-) \to CF^\bullet(L^+, h^+).$$

Example: Chekanov and Product tori

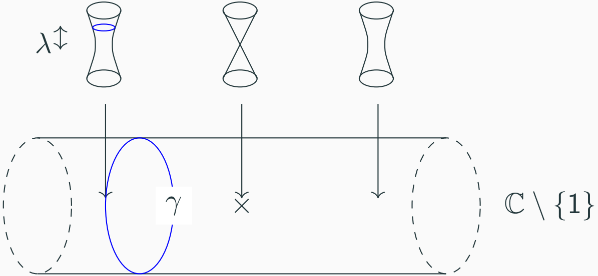
Lefschetz Fibration on $X = \mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$



We consider the Lefschetz fibration

$$\begin{aligned} W : \mathbb{C}^2 \setminus \{z_1 z_2 = 1\} &\to \mathbb{C} \setminus \{1\} \\ (z_1, z_2) &\mapsto z_1 z_2 \end{aligned}$$

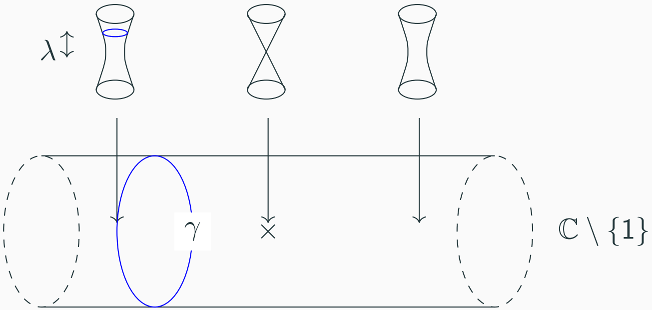
Lefschetz Fibration on $X = \mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$



To each loop $\gamma \in \mathbb{C} \setminus 1$ and $\lambda \in \mathbb{R}$ we construct a Lagrangian torus

$$L_{\gamma,\lambda} \in \mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$$

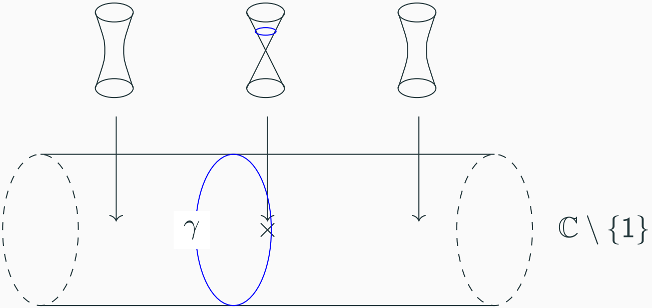
Lefschetz Fibration on $X = \mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$



When γ avoids the critical value zero the Lagrangian $L_{\gamma,\lambda}$ does not bound any holomorphic disks.

$$CF^\bullet(L_{\gamma,\lambda}) = CM^\bullet(T^2)$$
$$\mathcal{MC}(L_{\gamma,\lambda}) = H^1(T^2)$$

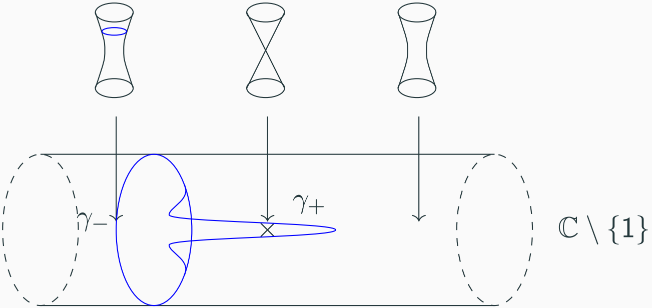
Lefschetz Fibration on $X = \mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$



Assume that $\lambda \neq 0$. If γ passes through 0, then $W^{-1}(0)$ intersects $L_{\gamma,\lambda}$ cleanly, giving a holomorphic disk.

Note that this is not a regular holomorphic disk.

Lefschetz Fibration on $X = \mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$

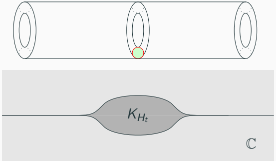


Let $\lambda \neq 0$. We now consider a family $L_{\gamma_t,\lambda}$ of Hamiltonian isotopic tori where one member passes through the critical fiber. We call:

$L_{\gamma_-,\lambda}$	Chekanov Type
$L_{\gamma_+,\lambda}$	Product Type

Picture of Holomorphic Disk

A holomorphic disk shows up in the middle of our cobordism:



Key Take away: If K is a cylindercial cobordism whose left end is unobstructed, then K is unobstructed.

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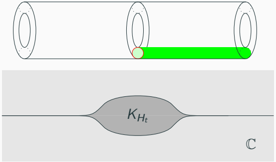

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Continuation Map

Let K_{H_t} be the Lagrangian suspension cobordism for $L_{\gamma_t,\lambda}$.

- We obtain a continuation map $\Theta_{H_t} : CF^\bullet(L_{\gamma_-,\lambda}) \to CF^\bullet(L_{\gamma_+,\lambda})$
- K_{H_t} bounds a *regular* Maslov index 0 holomorphic disk.
- Because we have map $i : CF^\bullet(L_{\gamma_-,\lambda}) \to CF^\bullet(K_{H_t})$, and the first is unobstructed, K_{H_t} is unobstructed.

Beyond Cylindrical Cobordisms

Monotone Lagrangian Cobordisms

While we exhibited continuation maps from Lagrangian cobordisms with topology $L \times \mathbb{R}$, but result should hold in more generality.

Theorem (Biran and Cornea 2013)

If $K : L^- \rightsquigarrow L^+$ is a monotone Lagrangian cobordism, then $CF^\bullet(L^-) \cong CF^\bullet(L^+)$ as chain complexes.

In fact, L^+, L^- are isomorphic as objects of the Fukaya category.

Question

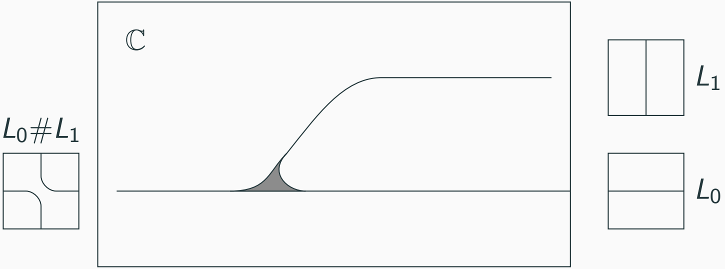
Are there any monotone 2-ended Lagrangian cobordisms besides Hamiltonian isotopy?

Lagrangian Surgery and Cobordism

Definition

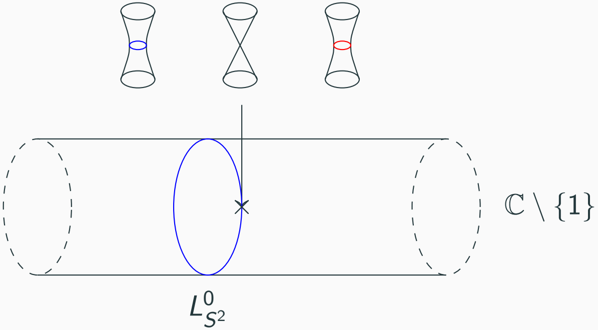
Let L^0, L^1 be Lagrangian submanifolds intersecting at a point p transversely. There is a *surgery cobordism*

$$K : (L_0, L_1) \rightsquigarrow L_0 \#_p L_1$$



Chekanov and Product tori

Let $X = \mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$. Pick γ which passes through the critical fiber, and $\lambda = 0$.



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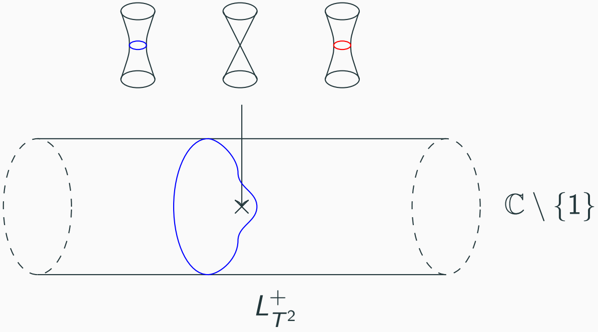

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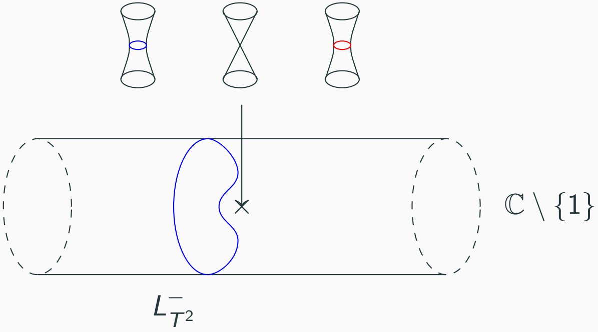


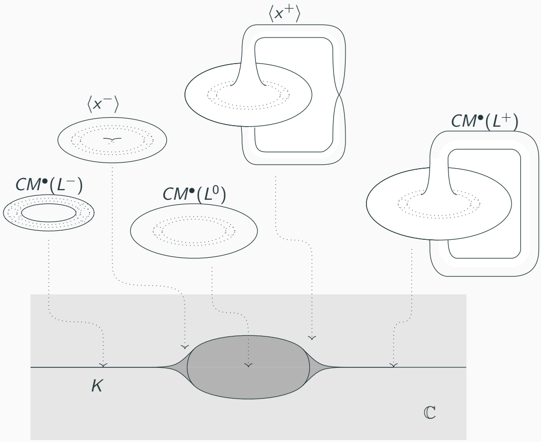
A Lagrangian cobordism.

So, there is a cobordism from the $L^+_{T^2}$ to the Whitney sphere, and then from the Whitney sphere to $L^-_{T^2}$. These can be concatenated to form a cobordism.

Claim (Haug 2015)

There exists an embedded Lagrangian cobordism K between $L^+_{T^2}$ and $L^-_{T^2}$, monotone tori of Product/Chekanov type.

Assembling the Cobordism



Comparison to Continuation Cobordism

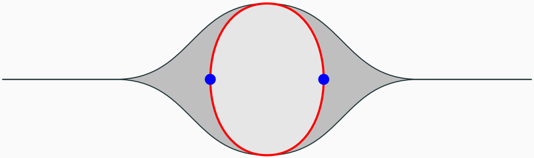
As before, we have projections from the cobordism to its end.

$$CF(L^-_{T^2}) \xleftarrow[\pi^-]{} CF^\bullet(K) \xrightarrow{\pi^+} CF^\bullet(L^+)$$

Problematically, $CM^\bullet(K)$ is not homotopic to $CM^\bullet(L^-_{T^2} \times \mathbb{R})$, so the argument from before will not carry over.

Holomorphic disks on K

The two surgery handles in this cobordism are what give it non-cylindrical topology.



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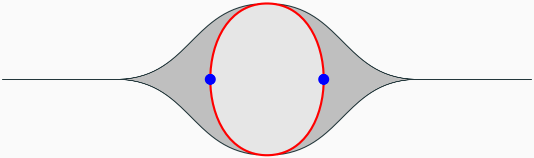
Claim

There exists a single injective holomorphic disk $u : (D^2, \partial D^2) \to (X \times T^*\mathbb{R}, K)$.

This holomorphic disk is characterized by $\pi_X \circ u(z) = (0, 0)$.

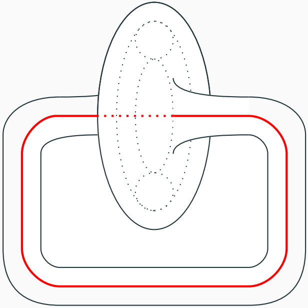


Bounding cochain for K

Claim

There exists a bounding cochain b on $CF^\bullet(K)$.

Idea of proof: The homology of $L^{\pm}_{T^2}$ jointly generate homology of K.

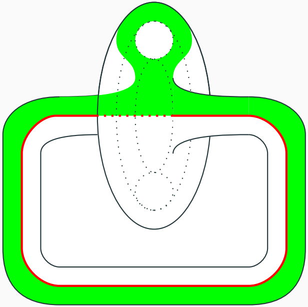


Bounding cochain for K

Claim

There exists a bounding cochain b on $CF^\bullet(K)$.

Idea of proof: The homology of $L^{\pm}_{T^2}$ jointly generate homology of K.



Returning to Continuation maps

When equipped with this bounding cochain, $CF^{\bullet}{}_b(K)$ fits into a new diagram:

$$CF^{\bullet}{}_{b^-}(L^-_{T^2}) \xleftarrow[\pi^-]{} CF^{\bullet}{}_b(K) \xrightarrow{\pi^+} CF^{\bullet}{}_{b^+}(L^+_{T^2})$$

This *does* give a mapping cylinder.

Theorem

There exists bounding cochains b^-, b^+ so that $(L^+_{T^2}, b^+), (L^-_{T^2}, b^-)$ are isomorphic in the Fukaya category.

- First example of non-cylindrical Lagrangian cobordism giving equivalence in Fukaya category.
- Similar to results in Rizell, Ekholm, and Tonkonog 2018; Pascaleff and Tonkonog 2020; Palmer and Woodward 2019, but result doesn't require us to know the structure of $\text{Fuk}(X)$.
- Why is this wall-crossing? The bounding cochains $b^\pm$ give rise to Local systems on $L^\pm_{T^2}$. These match the local systems associated to the Lagrangians under the wall-crossing transformation given in Auroux 2007.

Extensions of this result

- The proof that this Lagrangian cobordism gives an isomorphism of the ends is tailored to this specific example.
- There exist examples of $K : L^- \rightsquigarrow L^+$ with L^-, L^+ unobstructed, but never isomorphic for any deformation.

Ongoing Questions

It's likely that if the homology K is generated by homology of L^- and L^+, and L^- and L^+ are unobstructed, then K is unobstructed.

Conjecture

If every homology class of K is generated by homology of L^- and L^+ then there exists a deformation so that

$$CF^\bullet{}_{b^-}(L^-) \underset{\pi^-}{\longleftarrow} CF^\bullet{}_b(K) \xrightarrow{\pi^+} CF^\bullet{}_{b^+}(L^+)$$

is a mapping cylinder.

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